Qualitative features of periodic solutions of KdV

T. Kappeler*, B. Schaad†, P. Topalov‡

August 21, 2012

Abstract

In this paper we prove new qualitative features of solutions of KdV on the circle. The first result says that the Fourier coefficients of a solution of KdV in Sobolev space H^N, $N \geq 0$, admit a WKB type expansion up to first order with strongly oscillating phase factors defined in terms of the KdV frequencies. The second result provides estimates for the approximation of such a solution by trigonometric polynomials of sufficiently large degree.

Mathematics Subject Classification (2000): Primary: 35Q53, 35B40, 35B65. Secondary: 37K10, 37K15, 37K40

1 Introduction

Consider the Korteweg-de Vries equation (KdV)

$$\partial_t u = -\partial_x^3 u + 6u\partial_x u \tag{1}$$

on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. It is globally in time well-posed on the Sobolev spaces $H^N \equiv H^N(\mathbb{T}, \mathbb{R})$ with $N \geq -1$. The aim of this paper is to describe new qualitative features of periodic solutions of KdV. First note that in contrast to solutions on the real line, periodic solutions do not have a special profile decomposition as $t \to \pm\infty$. Our main point of interest, related to the numerical experiments of Fermi, Pasta, and Ulam of particle chains, is to know how the distribution of energy among the Fourier modes evolves. A partial result in this direction says that due to the integrals provided by the KdV hierarchy, the Sobolev norms of smooth solutions stay bounded uniformly in time. In this paper we make further contributions to the study of how the Fourier coefficients $\hat{u}_n(t) = \int_0^1 u(t,x)e^{-2\pi i n x}dx$ of a solution $u(t,x)$ of (1) evolve in time. Our first result aims at describing dispersion phenomena for solutions of KdV by studying

*Supported in part by the Swiss National Science Foundation
†Supported in part by the Swiss National Science Foundation
‡Supported in part by NSF DMS-0901443

how $\hat{u}_n(t)$ evolve for $|n|$ large. More precisely, we want to investigate if $\hat{u}_n(t)$ admits a WKB type expansion of the form

$$\hat{u}_n(t) = e^{iw_n t}\left(a_n(t) + \frac{b_n(t)}{n} + \dots\right), \tag{2}$$

where $e^{iw_n t}$ is a strongly oscillating phase factor with frequency w_n and the coefficients $a_n(t), b_n(t), \dots$ vary more slowly and satisfy the estimates

$$\sum n^{2N}|a_n(t)|^2 < \infty \quad \text{and} \quad \sum n^{2N}|b_n(t)|^2 < \infty.$$

To state our result more precisely, denote by ω_n, $n \geq 1$, the KdV frequencies of $u(t)$. Let us recall how they are defined. The KdV equation can be written as a Hamiltonian PDE with phase space L^2 and Poisson bracket

$$\{F, G\}(q) := \int_0^1 \partial F \, \partial_x \, \partial G dx \tag{3}$$

where F, G are C^1-functionals on L^2 and ∂F denotes the L^2-gradient of F. Then KdV takes the form $\partial_t u = \partial_x \partial_u \mathcal{H}$ where $\mathcal{H}$ is the KdV Hamiltonian

$$\mathcal{H}(q) := \int_0^1 \left(\frac{1}{2}(\partial_x q)^2 + q^3\right) dx.$$

In terms of this set-up, the ω_n's are given by

$$\omega_n = \partial_{I_n} \mathcal{H}.$$

Here we use that $\mathcal{H}$ can be expressed as a real analytic function of the action variables I_n, $n \geq 1$, so that the partial derivatives $\partial_{I_n}\mathcal{H}$ are well defined – see below for more details. Alternatively, ω_n can be viewed as a function of q, which by a slight abuse of terminology, we also denote by ω_n. Clearly, for any $n \geq 1$, $\omega_n(u(t))$ is independent of t and depends in a nonlinear fashion on $u(0)$. It is convenient to introduce

$$\omega_{-n} := -\omega_n \quad \forall n \in \mathbb{Z}_{\geq 1} \quad \text{and} \quad \omega_0 := 0$$

and to denote the KdV flow by S^t, i.e., $S^t(u(0)) = u(t)$. In addition, let

$$R^t(u(0)) := S^t(u(0)) - \sum_{n \in \mathbb{Z}} e^{i\omega_n t} \hat{u}_n(0) e^{2\pi i n x}$$

where for any $n \in \mathbb{Z}$, $\omega_n = \omega_n(u(0))$.

Theorem 1.1. *For $q = u(0) \in H^N$, $N \in \mathbb{Z}_{\geq 0}$, the error $R^t(q)$ of the approximation $\sum_{n\in\mathbb{Z}} e^{i\omega_n t}\hat{q}_n e^{2\pi i n x}$ of the flow $S^t(q)$ has the following properties:*

(i) $R^t : H^N \to H^{N+1}$ is continuous;

(ii) for any $q \in H^N$, the orbit $\{R^t(q) | \, t \in \mathbb{R}\}$ is relatively compact in H^{N+1};

(iii) for any $M > 0$, the set of orbits $\{R^t(q) | t \in \mathbb{R}, q \in H^N, \|q\|_{H^N} \leq M\}$ is bounded in H^{N+1};

(iv) if in addition $N \in \mathbb{Z}_{\geq 1}$, then $\partial_t R^t : H^N \to H^{N-1}$ is continuous and for any $q \in H^N$, the orbit $\{\partial_t R^t(q) | t \in \mathbb{R}\}$ is relatively compact in H^{N-1}. Moreover for any $M > 0$ the set of orbits $\{\partial_t R^t(q) | t \in \mathbb{R}, q \in H^N, \|q\|_{H^N} \leq M\}$ is bounded in H^{N-1}.

Remark 1.1. *Actually, one can prove that for any $c \in \mathbb{R}$, the restrictions of R^t and $\partial_t R^t$ to the affine subspace $H^N_c = \{q \in H^N | \int_0^1 q(x)dx = c\}$ are real analytic. See Remark 6.1 for a precise statement.*

Remark 1.2. *In case $u(0)$ is a finite gap potential, there are formulas, due to Its-Matveev [5], for the frequencies ω_n in terms of periods of an Abelian differential, defined on the spectral curve associated to $u(0)$. These formulas can be extended to potentials in H^N, $N \geq -1$, – cf [14]. Alternative formulas can be found in [10], Appendix F.*

Remark 1.3. *Note that the frequencies ω_n depend on the initial conditions in a nonlinear way. The statement of Theorem 1.1 no longer holds if the KdV frequencies ω_n are replaced by their linearization at 0, i.e., by $(2\pi n)^3$, confirming the belief of experts in the field that solutions of KdV on the circle are not approximated by linear evolution over a time interval of infinite length – see [2] for results on linear approximations of solutions over* finite *time intervals.*

In terms of the above WKB ansatz (2), Theorem 1.1 says that with $w_n := \omega_n$ and $a_n(t) := \hat{u}_n(0)$, the remainder term

$$\rho_n(t) := b_n(t) + \cdots := n \cdot \left(e^{-i\omega_n t}\hat{u}_n(t) - \hat{u}_n(0)\right) = n\widehat{R}^t_n(u(0))e^{-i\omega_n t}$$

satisfies $\sum n^{2N}|\rho_n(t)|^2 < \infty$ and in case $N \in \mathbb{Z}_{\geq 1}$,

$$\sum n^{2(N-2)}|\partial_t \rho_n(t)|^2 < \infty. \tag{4}$$

As the asymptotics of the KdV frequencies are given by $\omega_n = 8\pi^3 n^3 + O(n)$ (cf formula (97) in Section 6) estimate (4) quantifies the assertion that $(\rho_n(t))_{n\in\mathbb{Z}}$ varies more slowly than $(\hat{u}_n(t))_{n\in\mathbb{Z}}$.

The second result of this paper concerns the approximation of KdV solutions by trigonometric polynomials. For any $L \in \mathbb{Z}_{\geq 1}$, denote by $P_L : L^2 \to L^2$ the L^2-orthogonal projection of $L^2 = H^0(\mathbb{T}, \mathbb{R})$ onto the $2L+1$ dimensional $\mathbb{R}$-vector space generated by $e^{2\pi i n x}$, $|n| \leq L$.

Theorem 1.2. *Let $N \in \mathbb{Z}_{\geq 0}$ be arbitrary. Then for any $M > 0$ and $\epsilon > 0$ there exists $L_{\epsilon,M} \geq 1$ such that for any $u(0) \in H^N$, with $\|u(0)\|_{H^N} \leq M$, $L \geq L_{\epsilon,M}$ and any $t \in \mathbb{R}$*

$$\|(Id - P_L)u(0)\|_{H^N} - \epsilon \leq \|(Id - P_L)u(t)\|_{H^N} \leq \|(Id - P_L)u(0)\|_{H^N} + \epsilon.$$

In particular, if $u(0)$ with $\|u(0)\|_{H^N} \leq M$ is a trigonometric polynomial of order $L_\star$, then for any $L \geq \max(L_\star, L_{\epsilon,M})$, $P_L u(t)$ approximates $u(t)$ uniformly in $t \in \mathbb{R}$ up an error of size ϵ.

Remark 1.4. *The proof of Theorem 1.2 shows that for any* $|n| > L_{\epsilon,M}$ *and* $\|u(0)\|_{H^N} \leq M$,

$$|\hat{u}_n(0)| - \epsilon \leq |\hat{u}_n(t)| \leq |\hat{u}_n(0)| + \epsilon \quad \forall t \in \mathbb{R}.$$

It means that for $|n|$ *sufficiently large, the amplitude of the n'th Fourier mode is approximately constant, uniformly on bounded sets of* H^N.

Remark 1.5. *It follows from the proof of Theorem 1.1 that corresponding results hold for the flow of any Hamiltonian in the Poisson algebra of KdV. In particular, this is true for the flows of Hamiltonians in the KdV hierarchy.*

The main ingredient of the proofs of Theorem 1.1 and Theorem 1.2 are refined asymptotics of the Birkhoff map of KdV. This map provides normal coordinates, allowing to solve KdV by quadrature. Let us recall its set-up. First note that the average of any solution $u(t) \equiv u(t,x)$ of KdV in H^N is a conserved quantity. In particular, for any $c \in \mathbb{R}$, KdV leaves the subspaces $H_c^N \equiv H_c^N(\mathbb{T},\mathbb{R})$ of H^N invariant where

$$H_c^N = \left\{ p(x) = \sum \hat{p}_n e^{2\pi i n x} \mid \hat{p}_0 = c;\ \|p\|_N < \infty;\ \hat{p}_{-n} = \overline{\hat{p}_n}\ \forall n \in \mathbb{Z} \right\}$$

with

$$\|p\|_N := \left(\sum |n|^{2N} |\hat{p}_n|^2 \right)^{\frac{1}{2}}.$$

In the case $N = 0$, we often write L_c^2 for H_c^0 and $\|p\|$ instead of $\|p\|_0$. To describe the normal coordinates of KdV, let us introduce for any $\alpha \in \mathbb{R}$ the $\mathbb{R}$-subspace $\mathfrak{h}^\alpha$ of $\ell^{2,\alpha}$, given by

$$\mathfrak{h}^\alpha := \left\{ z = (z_n)_{n \neq 0} \in \ell^{2,\alpha} \mid z_{-n} = \overline{z}_n \forall n \geq 1 \right\}$$

where

$$\ell^{2,\alpha} \equiv \ell^{2,\alpha}(\mathbb{Z}_0, \mathbb{C}) := \left\{ z = (z_n)_{n \neq 0} \mid\ \|z\|_\alpha < \infty \right\},$$

$\mathbb{Z}_0 := \mathbb{Z} \setminus \{0\}$, and

$$\|z\|_\alpha := \left(\sum_{n \neq 0} |n|^{2\alpha} |z_n|^2 \right)^{\frac{1}{2}}.$$

The space $\mathfrak{h}^\alpha$ is endowed with the standard Poisson bracket for which $\{z_n, z_{-n}\} = -\{z_{-n}, z_n\} = 2i$ for any $n \geq 1$ whereas all other brackets between coordinate functions vanish. Furthermore we denote by $H_{0,\mathbb{C}}^N \equiv H_0^N(\mathbb{T},\mathbb{C})$, $L_{0,\mathbb{C}}^2 \equiv L_0^2(\mathbb{T},\mathbb{C})$ and $\mathfrak{h}_\mathbb{C}^\alpha$ the complexification of the spaces H_0^N, L_0^2, and $\mathfrak{h}^\alpha$. Note that $\mathfrak{h}_\mathbb{C}^\alpha = \ell^{2,\alpha}(\mathbb{Z}_0,\mathbb{C})$. A detailed proof of the following result can be found in [10] – cf also [8].

Theorem 1.3. *There exist an open neighbourhood* W *of* L_0^2 *in* $L_{0,\mathbb{C}}^2$ *and a real analytic map* $\Phi : W \to \mathfrak{h}_\mathbb{C}^{1/2}$ *with the following properties:*

(BC1) For any $N \in \mathbb{Z}_{\geq 0}$, *the restriction of* Φ *to* H_0^N *is a canonical, bianalytic diffeomorphism onto* $\mathfrak{h}^{N+1/2}$.

(BC2) *When expressed in the new coordinates, the KdV-Hamiltonian* $\mathcal{H} \circ \Phi^{-1}$, *defined on* $\mathfrak{h}^{3/2}$, *is a real analytic function of the action variables* $I_n = (z_n z_{-n})/2$, $n \geq 1$ *alone.*

(BC3) *The differential* $\Phi_0 \equiv d_0\Phi$ *of* Φ *at* 0 *is the weighted Fourier transform,*

$$\Phi_0(h) = \left(\frac{1}{\sqrt{|n|\pi}} \hat{h}_n \right)_{n \neq 0} \tag{5}$$

The coordinates z_n, $n \neq 0$, are referred to as (complex) Birkhoff coordinates whereas Φ is called Birkhoff map. Note that in [10] the Birkhoff map is defined slightly differently by setting $\Phi(q)$ to be $(x_n, y_n)_{n \geq 1}$ where $x_n = (z_n + z_{-n})/2$ and $y_n = i(z_n - z_{-n})/2$. The fact that KdV admits globally defined Birkhoff coordinates is a very special feature of KdV. In more physical terms it says that KdV, when considered with periodic boundary conditions, is a system of infinitely many coupled oscillators.

Remark 1.6. *A result similar to the one of Theorem 1.3 holds for the defocusing NLS equation. A detailed proof can be found in [4]. Cf also [20].*

The key ingredient of the proofs of Theorem 1.1 and Theorem 1.2 is the following result on the asymptotics of the Birkhoff map, which has an interest in its own.

Theorem 1.4. *For* $N \in \mathbb{Z}_{\geq 0}$, *there exists an open neighbourhood* W_N *of* H_0^N *in* $H_{0,\mathbb{C}}^N \cap W$ *so that* $\Phi - \Phi_0$ *maps* W_N *into* $\mathfrak{h}_{\mathbb{C}}^{N+3/2}$ *and, as a map from* W_N *to* $\mathfrak{h}_{\mathbb{C}}^{N+3/2}$, *is analytic. Here* W *is the neighbourhood of* L_0^2 *in* $L_{0,\mathbb{C}}^2$ *of Theorem 1.3. Furthermore, the restriction* $A := (\Phi - \Phi_0)_{|H_0^N} : H_0^N \to \mathfrak{h}^{N+3/2}$ *is a bounded map, i.e. it is bounded on bounded subsets of* H_0^N.

Remark 1.7. *In [23] a result similar to the one stated in Theorem 1.4 is proved for the Birkhoff map of KdV constructed in [7], where the phase space is endowed with the Poisson bracket introduced by Magri. As an application, a corresponding result is then derived for the modified Korteweg-de Vries equation (mKdV) on* H^N *with* $N \geq 1$. *Indeed, it was shown in [11] that the Miura map* $f \mapsto f' + f^2$ *canonically embeds the symplectic leaves of the phase space of mKdV, endowed with the Poisson bracket* (3), *into the phase space of KdV, endowed with the Magri bracket. (For a detailed study of the Miura map see [13].) As a consequence, results similar to the ones of Theorem 1.1 and Theorem 1.2 can be proved for mKdV – see [23].*

Remark 1.8. *We expect that similar results as the ones of Theorem 1.4 can be proved for the defocusing NLS equation. As a consequence, results similar to the ones of Theorem 1.1 and Theorem 1.2 are expected to hold for this equation.*

Remark 1.9. *The asymptotic estimates obtained to prove Theorem 1.4 can be used to derive a formula of the differential of the Birkhoff map* Φ *at* $q = 0$ *by*

a short calculation (cf Appendix A). In addition they can be used to get a short proof of the Fredholm property of the differential of Φ at any $q \in H_0^N$ (Corollary 5.2).

Remark 1.10. *Normalizing transformations such as the Birkhoff map are often viewed as nonlinear versions of the Fourier transform. In the case of KdV, Theorem 1.4 provides a qualitative statement in this respect, saying that Φ is a weakly nonlinear perturbation of the (weighted) Fourier transform.*

Related results Recently, Kuksin and Piatnitski initiated a study of random perturbations with damping of the KdV equation [18], [16]. More precisely they are interested, how the KdV-action variables evolve under certain perturbed equations. For this purpose they express the perturbed KdV equation in normal coordinates. Up to highest order, it is a linear differential equation if the nonlinear part $\Phi - \Phi_0$ of the Birkhoff map is 1-smoothing, i.e. if it maps H_0^N to $\mathfrak{h}^{N+3/2}$ for any $N \geq 0$. In their recent paper, Kuksin and Perelman [17] succeeded in showing that on a neighbourhood U of the equilibrium point $q = 0$, there exists a canonical, real analytic diffeomorphism $\Psi : U \to V$ with $V \subseteq \mathfrak{h}^{1/2}$ a neighbourhood of 0 in $\mathfrak{h}^{1/2}$ providing Birkhoff coordinates for KdV so that $\Psi - \Psi_0$ is 1-smoothing where Ψ_0 denotes the linearization of Ψ at $q = 0$ and coincides with Φ_0. They obtain the map Ψ by generalizing Eliasson's construction of a Birkhoff map near an equilibrium point of a finite dimensional integrable system to a class of integrable PDEs including the KdV equation. In order to apply Eliasson's construction, Kuksin and Perelman need coordinates for the KdV equation, provided in [6], as a starting point. Eliasson's construction is based on Moser's path-method and, in general, cannot be extended to get global coordinates. However, for the study of random perturbations of KdV in [16], global Birkhoff coordinates for KdV are needed. In [17], it was conjectured that there exists a globally defined Birkhoff map Ψ so that $\Psi - \Psi_0$ is 1-smoothing. Note that Birkhoff maps are not uniquely determined. Theorem 1.4 confirms that this conjecture holds true and that Ψ can be chosen to be the Birkhoff map of Theorem 1.3

The paper is organized as follows. In Section 2 we review asymptotic estimates of various spectral quantities, obtained in [12]. In Section 3 and Section 4, these estimates are used to improve on asymptotic estimates of actions, angles, and Birkhoff coordinates, obtained in [10]. In Section 5 we show Theorem 1.4 and in Section 6, Theorem 1.1 and Theorem 1.2 are proved.

If not stated otherwise we will use the notions and notations introduced in [10]. For the convenience of the reader we now recall the ones most frequently used in this paper. For q in $L^2_{0,\mathbb{C}}$, the Schrödinger operator $L_q := -d_x^2 + q$, considered on the interval $[0, 2]$ with periodic boundary conditions, has a discrete spectrum, consisting of a sequence of complex numbers bounded from below. We list them lexicographically and with algebraic multiplicities,

$$\lambda_0 \preceq \lambda_1 \preceq \lambda_2 \preceq \lambda_3 \preceq \dots$$

where two complex numbers a, b, are ordered lexicographically, $a \preceq b$, if $[\operatorname{Re} a <$

$\operatorname{Re} b$] or [$\operatorname{Re} a = \operatorname{Re} b$ and $\operatorname{Im} a \leq \operatorname{Im} b$]. These eigenvalues satisfy the asymptotics

$$(\lambda_{2n} - n^2\pi^2)_{n\geq 1}, (\lambda_{2n-1} - n^2\pi^2)_{n\geq 1} \in \ell^2$$

or, expressed in a more convenient form,

$$\lambda_{2n-1}, \lambda_{2n} = n^2\pi^2 + \ell_n^2,$$

valid uniformly on bounded subsets of $L^2_{0,\mathbb{C}}$. In particular, this means that for any $R > 0$ there exists $r > 0$ so that for any $q \in L^2_{0,\mathbb{C}}$ with $\|q\| \leq R$,

$$|\lambda_k - n^2\pi^2| \leq r\pi^2 \quad \forall k \in \{2n, 2n-1\}, \forall n \geq 1. \tag{6}$$

For real q, the periodic eigenvalues are real and satisfy

$$\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \dots .$$

Restricting ourselves to a sufficiently small neigbourhood W of L^2_0 in $L^2_{0,\mathbb{C}}$, we can always ensure that the closed intervals

$$G_n = \{(1-t)\lambda_{2n-1} + t\lambda_{2n} |\, 0 \leq t \leq 1\}, \quad n \geq 1,$$

as well as

$$G_0 = \{t + \lambda_0 |\, -\infty < t \leq 0\}$$

are disjoint from each other. By a slight abuse of terminology, for any $n \geq 1$, we refer to the closed interval G_n as the n'th gap and to $\gamma_n := \lambda_{2n} - \lambda_{2n-1}$, as the n'th gap length. We denote by τ_n the middle point of G_n, $\tau_n = (\lambda_{2n} + \lambda_{2n-1})/2$. Due to the asymptotic behaviour of the periodic eigenvalues, the G_n's admit mutually disjoint neighbourhoods $U_n \subseteq \mathbb{C}$ with $G_n \subseteq U_n$ called *isolating neighbourhoods*. Moreover, inside each U_n, we choose a circuit Γ_n around G_n with counterclockwise orientation. Both U_n and Γ_n can be chosen to be locally independent of q. For q in a sufficiently small neighbourhood W of L^2_0 in $L^2_{0,\mathbb{C}}$, the U_n's with $n \geq n_0$, $n_0 = n_0(q)$ sufficiently large, can be chosen to be discs, $U_n = \{\lambda \in \mathbb{C} |\, |\lambda - n^2\pi^2| < r\pi^2\}$, where $0 < r \leq n_0$ and n_0, r are chosen so large that (6) holds. Such neighbourhoods will be called isolating neighbourhoods with parameters $n_0 \geq 1$ and $r > 0$. In the course of this paper, W will be shrunk several times, but we continue to denote it by W.
By $\Delta(\lambda) \equiv \Delta(\lambda, q)$ we denote the discriminant of $-d_x^2 + q$,

$$\Delta(\lambda) = \operatorname{tr} M(1, \lambda)$$

where $M(x, \lambda)$ is the 2×2 matrix whose columns $(y_i(x,\lambda), y_i'(x,\lambda))^T$, $i = 1, 2$, are solutions of $-y'' + qy = \lambda y$ with $M(0, \lambda) = Id_{2\times 2}$. The function $\Delta(\lambda)$ is entire and $\Delta^2(\lambda) - 4$ has a product representation (see [10], Proposition B.10)

$$\Delta^2(\lambda) - 4 = 4(\lambda_0 - \lambda) \prod_{k\geq 1} \frac{(\lambda_{2k} - \lambda)(\lambda_{2k-1} - \lambda)}{\pi_k^4} \tag{7}$$

where $\pi_k = k\pi$ for any $k \geq 1$. For q in $L^2_{0,\mathbb{C}}$, we also need to consider the operator $-d_x^2 + q$ on $[0,1]$ with Dirichlet or Neumann boundary conditions. The corresponding spectra are again discrete, consisting of sequences of complex numbers, bounded from below. They are referred to as Dirichlet, respectively Neumann eigenvalues. We list them lexicographically and with their algebraic multiplicities

$$\mu_1 \preceq \mu_2 \preceq \mu_3 \ldots \quad \text{and} \quad \eta_0 \preceq \eta_1 \preceq \eta_2 \ldots .$$

They satisfy the asymptotics

$$\mu_n, \eta_n = n^2\pi^2 + \ell^2_n,$$

valid uniformly on bounded subsets of $L^2_{0,\mathbb{C}}$. For real q, the Dirichlet and the Neumann eigenvalues are real and satisfy

$$\lambda_1 \leq \mu_1 \leq \lambda_2 < \lambda_3 \leq \mu_2 \leq \lambda_4 < \ldots \quad \text{and} \quad \eta_0 \leq \lambda_0 < \lambda_1 \leq \eta_1 \leq \lambda_2 < \ldots .$$

Restricting ourselves to a sufficiently small neighbourhood W of L^2_0 in $L^2_{0,\mathbb{C}}$, we can assure that for any $q \in W$ there exist isolating neighbourhoods $U_n \subset \mathbb{C}$ so that $\mu_n, \eta_n \in U_n$, $n \geq 1$, whereas λ_0 and η_0 are not contained in any of the U_n's. Isolating neighbourhoods with this additional property can be chosen to be locally independent of q.
Finally let us recall the notion of the s-root, introduced in [10]. For $a, b \in \mathbb{C}$, we define on $\mathbb{C} \setminus \{(1-t)a + tb \,|\, 0 \leq t \leq 1\}$ the s-root of $(b-\lambda)(\lambda - a)$, determined by setting for $\lambda \in \mathbb{C}$ with $|\lambda - \tau| > |b - a|$

$$\sqrt[s]{(b-\lambda)(\lambda-a)} = i(\lambda - \tau)\sqrt[+]{1-w^2}$$

where $\tau = (b+a)/2$ and $w = (b-a)/2(\lambda - \tau)$ – see figure 2, p 62 in [10], showing a sign table. Here $\sqrt[+]{\lambda}$ denotes the principal branch of the square root on $\mathbb{C} \setminus (-\infty, 0]$ characterized by,

$$\sqrt[+]{\lambda} > 0 \quad \text{for } \lambda > 0.$$

Throughout the paper, $\log \lambda$ denotes the principal branch of the logarithm, defined on $\mathbb{C} \setminus (-\infty, 0]$. In particular, $\log 1 = 0$.

2 Prerequisites

In this section we review the asymptotic estimates of various spectral quantities, established in [12] which are needed for the proof of Theorem 1.4. The results concern the asymptotics of the Floquet exponents $(\kappa_n)_{n\geq 1}$, the Dirichlet eigenvalues $(\mu_n)_{n\geq 1}$, the Neumann eigenvalues $(\eta_n)_{n\geq 0}$, and the periodic eigenvalues $(\lambda_n)_{n\geq 0}$ of the Schrödinger operator $-d_x^2 + q$ for a potential q in $H^N_{0,\mathbb{C}}$ as well as the asymptotics of $\gamma_n^2 = (\lambda_{2n} - \lambda_{2n-1})^2$ and $\tau_n = (\lambda_{2n} + \lambda_{2n-1})/2$, $n \geq 1$. Recall that for q in $L^2_{0,\mathbb{C}}$,

$$\mu_n = n^2\pi^2 + \ell^2_n, \quad \eta_n = n^2\pi^2 + \ell^2_n \quad \text{and} \quad \lambda_{2n}, \lambda_{2n-1} = n^2\pi^2 + \ell^2_n.$$

For f, g in $L^2_{\mathbb{C}}$, let

$$\langle f, g\rangle = \int_0^1 f(x)\bar{g}(x)dx.$$

In particular, $\langle q, e^{2\pi ikx}\rangle = \int_0^1 q(x)e^{-2\pi ikx}dx$ denotes the k'th Fourier coefficient of q. The following asymptotics are known among experts in the field.

Theorem 2.1. *Let q be in $H^N_{0,\mathbb{C}}$ with $N \in \mathbb{Z}_{\geq 0}$. Then*

$$\mu_n = m_n - \langle q, \cos 2\pi nx\rangle + \frac{1}{n^{N+1}}\ell^2_n \tag{8}$$

$$\eta_n = m_n + \langle q, \cos 2\pi nx\rangle + \frac{1}{n^{N+1}}\ell^2_n \tag{9}$$

uniformly on bounded subsets of potentials in $H^N_{0,\mathbb{C}}$. The quantity m_n is of the form

$$m_n = n^2\pi^2 + \sum_{2\leq 2j\leq N+1} c_{2j}\frac{1}{n^{2j}} \tag{10}$$

with coefficients c_{2j} which are independent of n and N and given by integrals of polynomials in q and its derivatives up to order $2j-2$.

Remark 2.1. *The asymptotics* (8) *are proved in [19] and the uniform boundedness of the error in* (8) *is shown in [22]. A simple and self-contained proof of Theorem 2.1 can be found in [12].*

Using similar arguments as in the proof of Theorem 2.1 corresponding estimates for the periodic eigenvalues have been proved in [12].

Theorem 2.2. *Let q be in $H^N_{0,\mathbb{C}}$ with $N \in \mathbb{Z}_{\geq 0}$. Then*

$$\{\lambda_{2n}, \lambda_{2n-1}\} = \{m_n \pm \sqrt{\hat{q}_n\hat{q}_{-n} + \frac{1}{n^{2N+1}}\ell^2_n} + \frac{1}{n^{N+1}}\ell^2_n\} \tag{11}$$

uniformly on bounded subsets of potentials in $H^N_{0,\mathbb{C}}$. Again, m_n is the expression defined in (10)*.*

Remark 2.2. *A proof of the asymptotic estimate* (11)*, but not of the uniform boundedness of the error terms, can be found in [19].*

Unfortunately, the asymptotics of Theorem 2.2 do not suffice for our purposes. Actually we need estimates of $\gamma^2_n = (\lambda_{2n} - \lambda_{2n-1})^2$ and $\tau_n = (\lambda_{2n} + \lambda_{2n-1})/2$, $n \geq 1$ which are better than the ones obtained from Theorem 2.2.

Theorem 2.3. *Let $N \in \mathbb{Z}_{\geq 0}$. Then for any q in $H^N_{0,\mathbb{C}}$,*

$$\gamma^2_n = 4\hat{q}_n\hat{q}_{-n} + \frac{1}{n^{2N+1}}\ell^1_n \tag{12}$$

uniformly on bounded subsets in $H^N_{0,\mathbb{C}}$.

Proof. The claimed estimate follows from Theorem 1.2 in [9]. In the case at hand it says that

$$\min_{\pm} \left| \gamma_n \pm 2\sqrt{\rho(n)\rho(-n)} \right| = \frac{1}{n^{N+1}} \ell_n^2 \tag{13}$$

uniformly on bounded sets of $H^N_{0,\mathbb{C}}$. Here the sequence $(\rho(n))_{n \in \mathbb{Z}}$ is given by

$$\rho(n) := \langle q, e^{2\pi i n x} \rangle + \beta_1(n)$$

with

$$\beta_1(n) := \frac{1}{\pi^2} \sum_{k \neq \pm n} \frac{\langle q, e^{2\pi i (n-k)x} \rangle}{n-k} \frac{\langle q, e^{2\pi i (n+k)x} \rangle}{n+k}.$$

Then

$$\begin{aligned} &\left| \gamma_n^2 - 4\langle q, e^{2\pi i n x} \rangle \langle q, e^{-2\pi i n x} \rangle \right| \\ &\leq \left| \gamma_n^2 - 4\rho(n)\rho(-n) \right| + 4\left| \rho(n)\rho(-n) - \langle q, e^{2\pi i n x} \rangle \langle q, e^{-2\pi i n x} \rangle \right|. \end{aligned} \tag{14}$$

Note that

$$\gamma_n^2 - 4\rho(n)\rho(-n) = \left(\gamma_n + \epsilon_n 2\sqrt{\rho(n)\rho(-n)} \right) \left(\gamma_n - \epsilon_n 2\sqrt{\rho(n)\rho(-n)} \right),$$

where $\epsilon_n \in \{+, -\}$ is chosen such that

$$\left| \gamma_n + \epsilon_n 2\sqrt{\rho(n)\rho(-n)} \right| = \min_{\pm} \left| \gamma_n \pm 2\sqrt{\rho(n)\rho(-n)} \right|.$$

By Cauchy's inequality

$$\begin{aligned} &\sum_{n \geq 1} n^{2N+1} \left| \gamma_n^2 - 4\rho(n)\rho(-n) \right| \\ &\leq \left(\sum_{n \geq 1} n^{2N+2} \left| \gamma_n + \epsilon_n 2\sqrt{\rho(n)\rho(-n)} \right|^2 \right)^{1/2} \left(\sum_{n \geq 1} n^{2N} \left| \gamma_n - \epsilon_n 2\sqrt{\rho(n)\rho(-n)} \right|^2 \right)^{1/2}. \end{aligned}$$

By (13), the first factor of the latter product is uniformly bounded on bounded sets of q's in $W \cap H^N_{0,\mathbb{C}}$, whereas the second factor can be estimated by

$$\left(\sum_{n \geq 1} n^{2N} |\gamma_n|^2 \right)^{1/2} + 2\left(\sum_{n \geq 1} n^{2N} |\rho(n)|^2 \right)^{1/2} \left(\sum_{n \geq 1} n^{2N} |\rho(-n)|^2 \right)^{1/2}.$$

By [9], Theorem 1.1, $\left(\sum_{n \geq 1} n^{2N} |\gamma_n|^2 \right)^{1/2}$ is uniformly bounded on bounded sets of q's in W whereas by the definition of $\rho(\pm n)$

$$\left(\sum_{n \geq 1} n^{2N} |\rho(\pm n)|^2 \right)^{1/2} \leq \|q\|_N + \|\beta_1\|_N \leq \|q\|_N + \|q\|_N^2.$$

For the latter inequality we used that by [9], Lemma 2.10, $\|\beta_1\|_{N+1} \leq \|q\|_N^2$. It remains to estimate the second summand on the right hand side of (14). By the definition of $\rho(n)$

$$\begin{aligned}
&\sum_{n\geq 1} n^{2N+1} \left|\rho(n)\rho(-n) - \langle q, e^{2\pi i n x}\rangle\langle q, e^{-2\pi i n x}\rangle\right| \\
\leq& 2\|\beta_1\|_{N+1}\|q\|_N + \|\beta_1\|_{N+1}^2 \\
\leq& 2\|q\|_N^3(1 + \|q\|_N),
\end{aligned}$$

where we again used [9], Lemma 2.10. □

For the sequences $(\tau_n)_{n\geq 1}$ the following asymptotic estimates are proved in [12].

Theorem 2.4. *(i) For any $q \in H_0^N$, $N \in \mathbb{Z}_{\geq 0}$,*

$$\tau_n(q) = m_n + \frac{1}{n^{N+1}}\ell_n^2 \tag{15}$$

where m_n is given by (10) and the error term is uniformly bounded on bounded sets of potentials in H_0^N.

(ii) For any $N \in \mathbb{Z}_{\geq 0}$, there exists an open neighbourhood $W_N \subseteq H_{0,\mathbb{C}}^N$ of H_0^N so that (15) holds on W_N with a locally uniformly bounded error term.

Combining Theorem 2.1 and Theorem 2.4 one obtains

Corollary 2.1. *(i) For any $q \in H_0^N$, $N \in \mathbb{Z}_{\geq 0}$,*

$$\tau_n - \mu_n = \langle q, \cos 2\pi n x\rangle + \frac{1}{n^{N+1}}\ell_n^2 \tag{16}$$

where the error term is uniformly bounded on bounded sets of potentials in H_0^N.

(ii) For any $N \in \mathbb{Z}_{\geq 0}$, there exists an open neighbourhood $W_N \subseteq H_{0,\mathbb{C}}^N$ of H_0^N so that (16) holds on W_N with a locally uniformly bounded error term.

Furthermore we need asymptotic estimates for the Floquet exponents, defined by

$$\kappa_n = \log\left((-1)^n y_2'(1, \mu_n)\right). \tag{17}$$

Here $\mu_n = \mu_n(q)$ is the n'th Dirichlet eigenvalue of $L(q) = -d_x^2 + q$ with $q \in L_0^2$ and $y_2(x, \lambda)$ is the fundamental solution of $-y'' + qy = \lambda y$ satisfying $y_2(0, \lambda) = 0$ and $y_2'(0, \lambda) = 1$. Note that $y_2'(1, \mu_n) \neq 0$. Actually, it turns out that $(-1)^n y_2'(1, \mu_n) > 0$ for any $q \in L_0^2$. Hence $\log\left((-1)^n y_2'(1, \mu_n)\right)$ is well-defined with log denoting the principal branch of the logarithm. In fact, there exists a neighbourhood W of L_0^2 in $L_{0,\mathbb{C}}^2$ so that for $q \in W$ and any $n \geq 1$, $\kappa_n(q)$ is well-defined by (17). The κ_n's have been introduced in [3] and studied for square integrable potentials in [21]. Note that for $\lambda = \mu_n$ the Floquet matrix

$$M(\lambda) = \begin{pmatrix} y_1(1, \lambda) & y_2(1, \lambda) \\ y_1'(1, \lambda) & y_2'(1, \lambda) \end{pmatrix}$$

is lower triangular. Hence $y_2'(1,\mu_n)$ is one of the two Floquet multipliers of $M(\mu_n)$, the other one being $y_1(1,\mu_n)$ which by the Wronskian identity equals $1/y_2'(1,\mu_n)$. In particular it follows that

$$\kappa_n = -\log\left((-1)^n y_1(1,\mu_n)\right). \tag{18}$$

In [12] we prove

Theorem 2.5. *Let $N \geq 0$. Then for any q in $W \cap H^N_{0,\mathbb{C}}$,*

$$\kappa_n = \frac{1}{2\pi n}\left(\langle q, \sin 2\pi n x\rangle + \frac{1}{n^{N+1}}\ell^2_n\right)$$

uniformly on bounded subsets of $W \cap H^N_{0,\mathbb{C}}$.

Note that in contrast to the asymptotics of the Dirichlet eigenvalues or the Neumann eigenvalues, the size of κ_n for any $q \in H^N_0$ is of the order of $\frac{1}{n^{N+1}}\ell^2_n$. The case $N = 0$ is much simpler and has been treated in [21], p 60.
Finally we state some applications of the asymptotics of the periodic eigenvalues. For our purposes it suffices to consider potentials q in a sufficiently small neighbourhood W of L^2_0 in $L^2_{0,\mathbb{C}}$. Recall that we denote by $\Delta(\lambda)$ the discriminant of $-d_x^2 + q$ and that $\Delta^2(\lambda) - 4$ has the product representation

$$\Delta^2(\lambda) - 4 = 4(\lambda_0 - \lambda)\prod_{n\geq 1}\frac{(\lambda_{2n}-\lambda)(\lambda_{2n-1}-\lambda)}{\pi^4_n}$$

where $\pi_n = n\pi$ for any $n \geq 1$. Let $\dot{\Delta}(\lambda) = \partial_\lambda \Delta(\lambda)$. According to Proposition B.13 of [10] it also admits a product representation,

$$\dot{\Delta}(\lambda) = -\prod_{n\geq 1}\frac{\dot{\lambda}_n - \lambda}{\pi^2_n},$$

and the zeros $\dot{\lambda}_n$ satisfy

$$\dot{\lambda}_n - \tau_n = O(\gamma^2_n) \tag{19}$$

locally uniformly for q in W. Shrinking W, if necessary, we can assume without loss of generality that for any $n \geq 1$, $\dot{\lambda}_n \in U_n$ locally uniformly in W. The following estimate improves on the one of Propostition B.13 in [10].

Proposition 2.1. *For q in W,*

$$\dot{\lambda}_n - \tau_n = \frac{\gamma^2_n}{n}\ell^2_n \tag{20}$$

locally uniformly on W. On L^2_0, (20) *is uniformly bounded on bounded subsets of L^2_0.*

Proof. For any given $n \geq 1$, write

$$\Delta^2(\lambda) - 4 = \frac{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}{\pi_n^2} \Delta_n(\lambda) \tag{21}$$

where

$$\Delta_n(\lambda) = 4 \, \frac{\lambda - \lambda_0}{\pi_n^2} \left(\prod_{m \neq n} \frac{\lambda_{2m} - \lambda}{\pi_m^2} \right) \left(\prod_{m \neq n} \frac{\lambda_{2m-1} - \lambda}{\pi_m^2} \right). \tag{22}$$

Uniformly for $\lambda \in U_n$,

$$\frac{\lambda - \lambda_0}{\pi_n^2} = 1 + O\left(\frac{1}{n^2}\right) = 1 + \frac{1}{n} \ell_n^2.$$

By Corollary 7.1 in [12], uniformly for $\lambda \in U_n$,

$$\begin{aligned} \left(\prod_{m \neq n} \frac{\lambda_{2m} - \lambda}{\pi_m^2} \right) \left(\prod_{m \neq n} \frac{\lambda_{2m-1} - \lambda}{\pi_m^2} \right) &= \left(\frac{(-1)^{n+1}}{2} + \frac{1}{n} \ell_n^2 \right)^2 \\ &= \frac{1}{4} + \frac{1}{n} \ell_n^2, \end{aligned}$$

hence

$$\Delta_n(\lambda) = 1 + \frac{1}{n} \ell_n^2 \tag{23}$$

and by shrinking the size of the isolating neighbourhoods one obtains from Cauchy's estimate that

$$\dot{\Delta}_n(\lambda) = \frac{1}{n} \ell_n^2$$

uniformly in $n \geq 1$, $\lambda \in U_n$. By (21)

$$\begin{aligned} 0 &= \frac{d}{d\lambda} |_{\lambda = \dot{\lambda}_n} (\Delta^2(\lambda) - 4) \\ &= \frac{\lambda_{2n-1} - \dot{\lambda}_n + \lambda_{2n} - \dot{\lambda}_n}{\pi_n^2} \Delta_n(\dot{\lambda}_n) \\ &+ \frac{(\lambda_{2n} - \dot{\lambda}_n)(\dot{\lambda}_n - \lambda_{2n-1})}{\pi_n^2} \dot{\Delta}_n(\dot{\lambda}_n) \end{aligned}$$

or

$$0 = 2(\tau_n - \dot{\lambda}_n)(1 + \frac{1}{n} \ell_n^2) + \left(\frac{\gamma_n^2}{4} - (\tau_n - \dot{\lambda}_n)^2 \right) \frac{1}{n} \ell_n^2.$$

Hence

$$(\tau_n - \dot{\lambda}_n) \left(1 + \frac{1}{n} \ell_n^2 + (\tau_n - \dot{\lambda}_n) \frac{1}{n} \ell_n^2 \right) = \frac{\gamma_n^2}{n} \ell_n^2.$$

As by (19), $\tau_n - \dot{\lambda}_n = O(1)$ it then follows that

$$\tau_n - \dot{\lambda}_n = \frac{\gamma_n^2}{n}\ell_n^2$$

as claimed. Going through the arguments of the proof one sees that (20) holds locally uniformly on W and uniformly on bounded subsets of L_0^2. $\square$

By Theorem D.1 in [10] there exists a sequence $(\psi_n)_{n\geq 1}$ of entire functions,

$$\psi_n(\lambda) = \frac{2}{\pi_n}\prod_{m\neq n}\frac{\sigma_m^n - \lambda}{\pi_m^2} \tag{24}$$

so that

$$\frac{1}{2\pi}\int_{\Gamma_m}\frac{\psi_n(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}d\lambda = \delta_{mn} \quad \forall m \geq 1, \tag{25}$$

where $\sqrt[c]{\Delta^2(\lambda)-4}$ denotes the canonical root introduced in [10], section 6. Recall that Γ_m denotes a counterclockwise oriented circuit in U_m around the interval G_m and $\tau_m = (\lambda_{2m} + \lambda_{2m-1})/2$. By Theorem D.1 in [10], for any $n \geq 1$, the zeros σ_m^n, $m \neq n$, are real analytic functions of $q \in W$ so that $\sigma_m^n - \tau_m = O\left(\gamma_m^2/m\right)$ uniformly for $n \geq 1$, and locally uniformly in $q \in W$. Moreover one can choose W so that $\sigma_m^n \in U_m$ for any $n \geq 1$, $m \neq n$, and locally uniformly in W.

Proposition 2.2. *For q in W,*

$$\sigma_m^n - \tau_m = \frac{\gamma_m^2}{m}\ell_m^2 \tag{26}$$

locally uniformly on W and uniformly in n. On L_0^2, (26) *is uniformly bounded on bounded subsets of L_0^2.*

Proof. We drop the superscript in $\sigma^n = (\sigma_m^n)_{m\neq n}$ for the course of this proof. For $m \neq n$ one has

$$0 = \int_{\Gamma_m}\frac{\sigma_m - \lambda}{\sqrt[s]{(\lambda_{2m}-\lambda)(\lambda-\lambda_{2m-1})}}\chi_m^n(\sigma,\lambda)d\lambda \tag{27}$$

with

$$\chi_m^n(\sigma,\lambda) = (-1)^{m-1}\frac{\sigma_n - m^2\pi^2}{\sigma_n - \lambda}\frac{m\pi}{\sqrt[+]{\lambda-\lambda_0}}\prod_{j\neq m}\frac{\sigma_j - \lambda}{\sqrt[+]{(\lambda_{2j}-\lambda)(\lambda_{2j-1}-\lambda)}}$$

and $\sigma_n = \tau_n$. For $\lambda = m^2\pi^2 + \ell_m^2$, one has uniformly in $n \geq 1$, $m \neq n$,

$$\frac{\sigma_n - m^2\pi^2}{\sigma_n - \lambda} = 1 + \frac{\lambda - m^2\pi^2}{\sigma_n - \lambda} = 1 + \frac{1}{m}\ell_m^2$$

and $\sqrt[+]{\lambda - \lambda_0} = m\pi \sqrt[+]{1 + \frac{1}{m}\ell_m^2} = m\pi \left(1 + \frac{1}{m}\ell_m^2\right)$. Furthermore, by Proposition 7.1 in [12],

$$\left(\prod_{j \neq m} \frac{\sigma_j - \lambda}{\sqrt[+]{(\lambda_{2j} - \lambda)(\lambda_{2j-1} - \lambda)}} \right)^2 = 1 + \frac{1}{m}\ell_m^2. \tag{28}$$

Taking square roots on both sides of (28), one has

$$(-1)^{m-1} \prod_{j \neq m} \frac{\sigma_j - \lambda}{\sqrt[+]{(\lambda_{2j} - \lambda)(\lambda_{2j-1} - \lambda)}} = 1 + \frac{1}{m}\ell_m^2.$$

Altogether we have for $\lambda = m^2\pi^2 + \ell_m^2$, $\lambda \in U_m$,

$$\chi_m^n(\lambda) = 1 + \frac{1}{m}\ell_m^2 \tag{29}$$

uniformly in n. The integral

$$\int_{\Gamma_m} \frac{\sigma_m - \lambda}{\sqrt[s]{(\lambda_{2m} - \lambda)(\lambda - \lambda_{2m-1})}} d\lambda \tag{30}$$

can be explicitly computed. In the case where $\lambda_{2m} = \lambda_{2m-1}$, one gets from the definition of the s-root and Cauchy's formula that the integral equals $2\pi(\sigma_m - \tau_m)$. In the case $\lambda_{2m} \neq \lambda_{2m-1}$,

$$\int_{\Gamma_m} \frac{\sigma_m - \lambda}{\sqrt[s]{(\lambda_{2m} - \lambda)(\lambda - \lambda_{2m-1})}} d\lambda = 2 \int_{\lambda_{2m-1}}^{\lambda_{2m}} \frac{\sigma_m - \lambda}{\sqrt[+]{(\lambda_{2m} - \lambda)(\lambda - \lambda_{2m-1})}} d\lambda.$$

By the change of coordinate $\lambda = \tau_m + t\frac{\gamma_m}{2}$ we get

$$\begin{aligned}\int_{\Gamma_m} \frac{\sigma_m - \lambda}{\sqrt[s]{(\lambda_{2m} - \lambda)(\lambda - \lambda_{2m-1})}} d\lambda &= 2 \int_{-1}^{1} \frac{\sigma_m - \tau_m}{\sqrt[+]{1 - t^2}} dt - \gamma_m \int_{-1}^{1} \frac{t}{\sqrt[+]{1 - t^2}} dt \\ &= 2(\sigma_m - \tau_m) \int_{-1}^{1} \frac{1}{\sqrt[+]{1 - t^2}} dt.\end{aligned}$$

Therefore in both cases

$$\frac{1}{2\pi} \int_{\Gamma_m} \frac{\sigma_m - \lambda}{\sqrt[s]{(\lambda_{2m} - \lambda)(\lambda - \lambda_{2m-1})}} d\lambda = \sigma_m - \tau_m.$$

Hence, by (27)

$$0 = (\sigma_m - \tau_m)\chi_m^n(\tau_m) + \frac{1}{2\pi} \int_{\Gamma_m} \frac{(\sigma_m - \lambda)(\chi_m^n(\lambda) - \chi_m^n(\tau_m))}{\sqrt[s]{(\lambda_{2m} - \lambda)(\lambda - \lambda_{2m-1})}} d\lambda. \tag{31}$$

As $\tau_m = m^2\pi^2 + \ell_m^2$ it follows from (29) that

$$\chi_m^n(\tau_m) = 1 + \frac{1}{m}\ell_m^2 \tag{32}$$

and, by the Taylor expansion of χ_m^n at τ_m and Cauchy's estimate, for $\lambda = m^2\pi^2 + \ell_m^2$, $\lambda \in U_m$

$$\frac{\chi_m^n(\lambda) - \chi_m^n(\tau_m)}{\lambda - \tau_m} = \frac{1}{m}\ell_m^2.$$

Finally as $\sigma_m - \lambda = O(\gamma_m), \lambda \in G_m$, one has $(\sigma_m - \lambda)(\lambda - \tau_m) = O(\gamma_m^2)$ for $\lambda \in G_m$. Hence by Lemma M.1 in [10]

$$\begin{aligned}
&\int_{\Gamma_m} \frac{(\sigma_m - \lambda)(\chi_m^n(\lambda) - \chi_m^n(\tau_m))}{\sqrt[s]{(\lambda_{2m} - \lambda)(\lambda - \lambda_{2m-1})}} d\lambda \\
=&2\int_{G_m} \frac{(\sigma_m - \lambda)(\lambda - \tau_m)\frac{\chi_m^n(\lambda) - \chi_m^n(\tau_m)}{\lambda - \tau_m}}{\sqrt[+]{(\lambda_{2m} - \lambda)(\lambda - \lambda_{2m-1})}} d\lambda \\
=&\frac{\gamma_m^2}{m}\ell_m^2
\end{aligned}$$

This together with (31) gives

$$\sigma_m^n - \tau_m = \frac{\gamma_m^2}{m}\ell_m^2$$

uniformly in n. Going through the arguments of the proof one sees that (26) holds locally uniformly on W and uniformly on bounded subsets of L_0^2. $\square$

3 Asymptotics of actions and angles

In this section we improve on estimates of the actions and angles obtained in [10], section 7 respectively section 8. Let us begin with asymptotics of the actions. Recall that W is a (sufficiently small) neighbourhood of L_0^2 in $L_{0,\mathbb{C}}^2$. For q in W, the n'th action variable is defined by

$$I_n = \frac{1}{\pi}\int_{\Gamma_n} \lambda \frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\Delta^2(\lambda) - 4}} d\lambda \qquad (n \geq 1),$$

where $\Delta(\lambda)$ is the discriminant, $\dot{\Delta}(\lambda) = \partial_\lambda \Delta(\lambda)$ and Γ_n is a contour around the interval $G_n := \{t\lambda_{2n-1} + (1-t)\lambda_{2n} |\ 0 \leq t \leq 1\}$ in the isolating neighbourhood U_n of G_n – see at the end of the introduction or section 7 of [10] for more details. First we need to derive improved estimates of the quotient I_n/γ_n^2, given in [10], Theorem 7.3.

Proposition 3.1. *Locally uniformly on W, the quotient I_n/γ_n^2 satisfies*

$$8\pi n \frac{I_n}{\gamma_n^2} = 1 + \frac{1}{n}\ell_n^2. \tag{33}$$

Moreover

$$\xi_n = \sqrt[+]{8I_n/\gamma_n^2} = \frac{1}{\sqrt{\pi n}}(1 + \frac{1}{n}\ell_n^2) \tag{34}$$

is well-defined as a real analytic, non-vanishing function on W. In particular, at $q = 0$, we have $\xi_n = \frac{1}{\sqrt{\pi n}}$ for all $n \geq 1$. On L_0^2, (34) holds uniformly on bounded subsets of L_0^2.

Proof. We refer to [10], section 7, for all notions, notations (and results) not explained here. In view of Theorem 7.3 in [10] it only remains to be shown the improved asymptotics (33). Recall the product expansions

$$\Delta^2(\lambda) - 4 \quad = \quad 4(\lambda_0 - \lambda) \prod_{n \geq 1} \frac{(\lambda_{2n} - \lambda)(\lambda_{2n-1} - \lambda)}{\pi_n^4}$$

and

$$\dot{\Delta}(\lambda) \quad = \quad -\prod_{n \geq 1} \frac{\dot{\lambda}_n - \lambda}{\pi_n^2}.$$

For λ on Γ_n write

$$\frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\Delta^2(\lambda) - 4}} = \frac{1}{2\pi n} \frac{\lambda - \dot{\lambda}_n}{\sqrt[s]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} \chi_n(\lambda) \tag{35}$$

where for $\lambda \in U_n$,

$$\chi_n(\lambda) = (-1)^{n-1} \frac{n\pi}{\sqrt[+]{\lambda - \lambda_0}} \prod_{m \neq n} \frac{\dot{\lambda}_m - \lambda}{\sqrt[+]{(\lambda_{2m} - \lambda)(\lambda_{2m-1} - \lambda)}}, \tag{36}$$

and where the canonical root $\sqrt[c]{\Delta^2(\lambda) - 4}$ has been introduced earlier. (For questions of signs, see section 6 in [10].) Note that for $\lambda \in U_n$,

$$\frac{n\pi}{\sqrt[+]{\lambda - \lambda_0}} = 1 + \frac{1}{n} \ell_n^2. \tag{37}$$

Furthermore, by Proposition 2.1, the roots $\dot{\lambda}_n$ of $\dot{\Delta}$ satisfy $\dot{\lambda}_n = \tau_n + \frac{\gamma_n^2}{n} \ell_n^2$ and hence, by [12], Proposition 2.1, one has for $\lambda \in U_n$,

$$\begin{aligned} \left(\prod_{m \neq n} \frac{\dot{\lambda}_m - \lambda}{\sqrt[+]{(\lambda_{2m} - \lambda)(\lambda_{2m-1} - \lambda)}} \right)^2 &= \prod_{m \neq n} \frac{\dot{\lambda}_m - \lambda}{\lambda_{2m} - \lambda} \prod_{m \neq n} \frac{\dot{\lambda}_m - \lambda}{\lambda_{2m-1} - \lambda} \\ &= 1 + \frac{1}{n} \ell_n^2 \end{aligned}$$

or

$$(-1)^{n-1} \prod_{m \neq n} \frac{\dot{\lambda}_m - \lambda}{\sqrt[+]{(\lambda_{2m} - \lambda)(\lambda_{2m-1} - \lambda)}} = 1 + \frac{1}{n} \ell_n^2. \tag{38}$$

Combining (37) and (38) leads to the estimate

$$\chi_n(\lambda) = 1 + \frac{1}{n}\ell_n^2 \tag{39}$$

uniformly for $\lambda \in U_n$. Introduce

$$Z_n = \{q \in W | \; \lambda_{2n}(q) = \lambda_{2n-1}(q)\}.$$

Arguing as in section 7 of [10] one gets for q in $W \setminus Z_n$

$$\begin{aligned} I_n &= \frac{1}{\pi}\int_{\Gamma_n} (\lambda - \dot{\lambda}_n)\frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}d\lambda \\ &= \frac{1}{2\pi^2 n}\int_{\Gamma_n} \frac{(\lambda-\dot{\lambda}_n)^2}{\sqrt[s]{(\lambda_{2n}-\lambda)(\lambda-\lambda_{2n-1})}}\chi_n(\lambda)d\lambda. \end{aligned}$$

Substituting $\lambda = \tau_n + \zeta\gamma_n/2$ and setting $\delta_n = 2(\dot{\lambda}_n - \tau_n)/\gamma_n$ yields

$$I_n = \frac{\gamma_n^2}{8\pi^2 n}\int_{\Gamma_n'} (\zeta-\delta_n)^2 \chi_n(\tau_n + \zeta\frac{\gamma_n}{2})\frac{d\zeta}{\sqrt[s]{1-\zeta^2}}$$

where Γ_n' is some circuit in $\mathbb{C}$ around $[-1,1]$. Thus on $W \setminus Z_n$,

$$\begin{aligned} 8\pi n\frac{I_n}{\gamma_n^2} &= \frac{1}{\pi}\int_{\Gamma_n'} (\zeta-\delta_n)^2\chi_n\left(\tau_n + \zeta\frac{\gamma_n}{2}\right)\frac{d\zeta}{\sqrt[s]{1-\zeta^2}} \\ &= \frac{1}{\pi}\int_{\Gamma_n'} (\zeta-\delta_n)^2\chi_n(\tau_n)\frac{d\zeta}{\sqrt[s]{1-\zeta^2}} \\ &+ \frac{1}{\pi}\int_{\Gamma_n'} (\zeta-\delta_n)^2\left(\chi_n\left(\tau_n+\zeta\frac{\gamma_n}{2}\right) - \chi_n(\tau_n)\right)\frac{d\zeta}{\sqrt[s]{1-\zeta^2}}. \end{aligned}$$

By Proposition 2.1, δ_n is well defined on all of W, hence so is the r.h.s. of the latter identity. As

$$(\zeta-\delta_n)^2 = \zeta^2 + \frac{\gamma_n}{n}\ell_n^2$$

Lemma M.1 in [10] and (39) then imply that

$$\begin{aligned} \frac{1}{\pi}\int_{\Gamma_n'} (\zeta-\delta_n)^2\chi_n(\tau_n)\frac{d\zeta}{\sqrt[s]{1-\zeta^2}} &= \frac{1}{\pi}\int_{\Gamma_n'} \frac{\zeta^2 d\zeta}{\sqrt[s]{1-\zeta^2}} + \frac{1}{n}\ell_n^2 \\ &= 1 + \frac{1}{n}\ell_n^2. \end{aligned}$$

By the Taylor expansion of order 0 of χ_n at $\lambda = \tau_n$, Cauchy's estimate, and (39) to bound $\dot{\chi}_n(\lambda)$, one gets

$$\frac{1}{\pi}\int_{\Gamma_n'} (\zeta-\delta_n)^2\left(\chi_n(\tau_n+\zeta\frac{\gamma_n}{2}) - \chi_n(\tau_n)\right)\frac{d\zeta}{\sqrt[s]{1-\zeta^2}} = \frac{1}{n}\ell_n^2.$$

Altogether,

$$8\pi n I_n/\gamma_n^2 = 1 + \frac{1}{n}\ell_n^2$$

and thus

$$\xi_n = \frac{1}{\sqrt{\pi n}}(1 + \frac{1}{n}\ell_n^2).$$

Going through the arguments of the proof one sees that the estimate (34) holds locally uniformly on W and uniformly on bounded subsets of L_0^2. $\square$

Proposition 3.1 leads to the following asymptotics of the action variables.

Proposition 3.2. *Locally uniformly on $W \cap H^N_{0,\mathbb{C}}$*

$$2I_n = \frac{1}{\pi n}\left(\hat{q}_n\hat{q}_{-n} + \frac{1}{n^{2N+1}}\ell_n^1\right). \tag{40}$$

On H_0^N, the error in (40) is uniformly bounded on bounded subsets of H_0^N.

Proof. By Theorem 2.3 one has

$$\gamma_n^2 = 4\hat{q}_n\hat{q}_{-n} + \frac{1}{n^{2N+1}}\ell_n^1.$$

Combined with the asymptotics (33) we then get,

$$2I_n = \frac{1}{4n\pi}\gamma_n^2\left(1 + \frac{1}{n}\ell_n^2\right) = \frac{1}{\pi n}(1 + \frac{1}{n}\ell_n^2)\left(\hat{q}_n\hat{q}_{-n} + \frac{1}{n^{2N+1}}\ell_n^1\right)$$

or

$$2I_n = \frac{1}{\pi n}\left(\hat{q}_n\hat{q}_{-n} + \frac{1}{n^{2N+1}}\ell_n^1\right).$$

Going through the arguments of the proof one sees that (40) holds locally uniformly on $W \cap H^N_{0,\mathbb{C}}$ and uniformly on bounded subsets of H_0^N. $\square$

Next we improve on the estimates of the angle variables obtained in [10], section 8. To this end we use the improved estimate of the zeros $(\sigma_m^n)_{m\neq n}$ of the entire function ψ_n,

$$\sigma_m^n = \tau_m + \frac{\gamma_m^2}{m}\ell_m^2$$

of Proposition 2.2. Recall that $Z_n = \{q \in W \mid \gamma_n(q) = 0\}$ where W is the neighbourhood of L_0^2 in $L^2_{0,\mathbb{C}}$ of Theorem 1.3. For $q \in W \setminus Z_n$ denote, as in [10], by $\theta_n(q)$ the n'th angle variable

$$\theta_n(q) = \beta_{n,n}(q) + \beta_n(q) \quad \text{and} \quad \beta_n(q) = \sum_{k\neq n}\beta_{n,k}(q)$$

where

$$\beta_{n,n}(q) = \int_{\lambda_{2n-1}}^{\mu_n^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}} d\lambda \quad \mathrm{mod} 2\pi, \tag{41}$$

and, for $k \neq n$,

$$\beta_{n,k}(q) = \int_{\lambda_{2k-1}}^{\mu_k^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}} d\lambda. \tag{42}$$

Here μ_n^* is the point $(\mu_n, \sqrt[*]{\Delta^2(\mu_n) - 4})$ on the affine curve Σ_q,

$$\Sigma_q = \left\{(\lambda, z) \in \mathbb{C}^2 | z^2 = \Delta^2(\lambda) - 4\right\}, \tag{43}$$

with $\mu_n = \mu_n(q)$ denoting the n'th Dirichlet eigenvalue of $-d_x^2 + q$ and $\sqrt[*]{\Delta^2(\mu_n) - 4}$ denoting the square root of $\Delta^2(\mu_n) - 4$ given by

$$\sqrt[*]{\Delta^2(\mu_n) - 4} = y_1(1, \mu_n) - y_2'(1, \mu_n). \tag{44}$$

The integral in (41) is a straight line integral from $(\lambda_{2n-1}, 0)$ to μ_n^* in Σ_q and the one in (42) is defined similarly. See section 6 in [10] for more explanations concerning the notation. Let us begin by analyzing $\beta_{n,k}$ in more detail. In Lemma 8.2 and Lemma 8.3 of [10], it is shown that $\beta_{n,k}$ $(k \neq n)$ is a well defined analytic function on W. In Theorem 8.5 of [10], it is shown that $\beta_n = \sum_{k \neq 0} \beta_{n,k}$ is absolutely summable on W and with the help of Lemma 8.4 in [10] one proves that the following estimate holds.

Lemma 3.1. *Locally uniformly on W,*

$$\beta_n = O\left(\frac{1}{n}\right). \tag{45}$$

On L_0^2, (45) holds uniformly on bounded subsets of L_0^2.

Next let us consider $\beta_{n,n}$ in more detail. Recall that $\beta_{n,n}$ is defined on $W \setminus Z_n$ mod 2π and is analytic on $W \setminus Z_n$ mod π. Similarly as in (35), we write for λ near G_n

$$\frac{\psi_n(\lambda)}{\sqrt[c]{\Delta(\lambda)^2 - 4}} = \frac{\zeta_n(\lambda)}{\sqrt[s]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} \tag{46}$$

where

$$\zeta_n(\lambda) = (-1)^{n-1} \frac{\pi n}{\sqrt[+]{\lambda - \lambda_0}} \prod_{m \neq n} \frac{\sigma_m^n - \lambda}{\sqrt[+]{(\lambda_{2m} - \lambda)(\lambda_{2m-1} - \lambda)}}. \tag{47}$$

The following results improve on the asymptotic estimates of Lemma 9.2 in [10].

Lemma 3.2. *(i) Uniformly for $\lambda \in U_n$*

$$\zeta_n(\lambda) = 1 + \frac{1}{n}\ell_n^2.$$

(ii) Uniformly for $\mu \in G_n$

$$\zeta_n(\mu) = 1 + \frac{\gamma_n}{n}\ell_n^2.$$

(iii) The error estimates in (i) and (ii) hold locally uniformly on W and are uniformly bounded on bounded subsets of L_0^2.

Proof. (i) Note that by [12], Proposition 2.1

$$(-1)^{n-1} \prod_{m \neq n} \frac{\sigma_m^n - \lambda}{\sqrt[+]{(\lambda_{2m} - \lambda)(\lambda_{2m-1} - \lambda)}} = 1 + \frac{1}{n}\ell_n^2 \tag{48}$$

uniformly for $\lambda \in U_n$. With

$$\lambda - \lambda_0 = n^2\pi^2 \left(1 + \frac{\lambda - n^2\pi^2 - \lambda_0}{n^2\pi^2}\right)$$

one gets

$$\frac{\sqrt[+]{\lambda - \lambda_0}}{n\pi} = \sqrt[+]{1 + \frac{\lambda - n^2\pi^2 - \lambda_0}{n^2\pi^2}} = 1 + O\left(\frac{1}{n^2}\right). \tag{49}$$

Together, (48) and (49) yield

$$\zeta_n(\lambda) = 1 + \frac{1}{n}\ell_n^2 \tag{50}$$

uniformly for $\lambda \in U_n$.

(ii) Assume that $\gamma_n \neq 0$. By the normalisation of the ψ_n-function,

$$\int_{\lambda_{2n-1}}^{\lambda_{2n}} \frac{\psi_n(\lambda)}{\sqrt[c]{\Delta(\lambda)^2 - 4}} d\lambda = \pi$$

for the line integral from λ_{2n-1} to λ_{2n} obtained by deforming Γ_n to the interval $[\lambda_{2n-1}, \lambda_{2n}]$. By (46) one then gets for any $\mu \in U_n$

$$\begin{aligned}
\pi &= \int_{\lambda_{2n-1}}^{\lambda_{2n}} \frac{\zeta_n(\lambda)}{\sqrt[s]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda \\
&= \int_{\lambda_{2n-1}}^{\lambda_{2n}} \frac{\zeta_n(\mu)}{\sqrt[s]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda + \int_{\lambda_{2n-1}}^{\lambda_{2n}} \frac{\zeta_n(\lambda) - \zeta_n(\mu)}{\sqrt[s]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda.
\end{aligned}$$

By a straightforward computation,

$$\int_{\lambda_{2n-1}}^{\lambda_{2n}} \frac{d\lambda}{\sqrt[s]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} = \pi. \tag{51}$$

Combined with Lemma M.1 in [10], we then obtain

$$|\zeta_n(\mu) - 1| = \frac{1}{\pi}\left|\int_{\lambda_{2n-1}}^{\lambda_{2n}} \frac{\zeta_n(\lambda) - \zeta_n(\mu)}{\sqrt[s]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda\right|$$
$$\leq \max_{\lambda \in G_n} |\zeta_n(\lambda) - \zeta_n(\mu)|. \tag{52}$$

This bound holds no matter if $\gamma_n \neq 0$ or not. Recall that $\zeta_n(\lambda)$ is analytic for λ in U_n. Hence one concludes from (50) and Cauchy's estimate that

$$\dot{\zeta}_n(\mu) = \frac{1}{n}\ell_n^2$$

uniformly for $\mu \in U_n$ and thus by the mean value theorem, for $q \in W$ and $\mu \in G_n$,

$$\max_{\lambda \in G_n} |\zeta_n(\lambda) - \zeta_n(\mu)| \leq \frac{|\gamma_n|}{n}\ell_n^2$$

uniformly for $\mu \in G_n$ as claimed. This combined with (52) shows (ii).

(iii) Going through the arguments of the proofs of (i) and (ii) one sees that the estimates hold locally uniformly on W and uniformly on bounded subsets of L_0^2.

□

Instead of describing the improved asymptotics of $\beta_{n,n}$, we directly study the asymptotics of $z_n^{\pm}$, defined on $W \setminus Z_n$ by

$$z_n^{\pm} = \gamma_n e^{\pm i\beta_{n,n}}. \tag{53}$$

This is the topic of the following section.

4 Asymptotics of $z_n^{\pm}$

The purpose of this section is to prove sharp asymptotic estimates of $z_n^{\pm}$. First note that it was shown in [10] that $z_n^{\pm}$, given by (53), analytically extend to all of W – see formula (9.4) in [10].

Proposition 4.1. *For $N \in \mathbb{Z}_{\geq 0}$, let $W_N \subseteq W \cap H_{0,\mathbb{C}}^N$ be the neighbourhood of H_0^N given in Theorem 2.4. For any $q \in W_N$,*

$$z_n^{\pm} = 2\hat{q}_{\mp n} + \frac{1}{n^{N+1}}\ell_n^2$$

locally uniformly on $W \cap H_{0,\mathbb{C}}^N$. On H_0^N, the error is uniformly bounded on bounded subsets of H_0^N.

First we need to make some preparations. Assume that $q \in W \setminus Z_n$. Then

$$z_n^{\pm} = \gamma_n \exp\left(\pm i \int_{\lambda_{2n-1}}^{\mu_n^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}} d\lambda\right). \tag{54}$$

If $\mu_n = \lambda_{2n-1}$, then $z_n^\pm = \gamma_n$ whereas if $\mu_n = \lambda_{2n}$, then $z_n^\pm = -\gamma_n$ by the normalization of ψ_n. By Lemma 9.1 in [10], $z_n^\pm$ are analytic functions on $W \setminus Z_n$. In the case where $\mu_n \notin \{\lambda_{2n}, \lambda_{2n-1}\}$, we choose as path of integration the interval $[\lambda_{2n-1}, \mu_n]$ in $\mathbb{C}$ and obtain the following formula

$$z_n^\pm = \gamma_n \exp\left(\pm i \int_{\lambda_{2n-1}}^{\mu_n} \frac{\psi_n(\lambda)}{\sqrt[*]{\Delta(\lambda)^2 - 4}} d\lambda \right) \tag{55}$$

where for λ in $[\lambda_{2n-1}, \mu_n]$, $\sqrt[*]{\Delta(\lambda)^2 - 4}$ is defined to be the continuous function on $[\lambda_{2n-1}, \lambda_{2n}]$ with sign determined by

$$\sqrt[*]{\Delta(\lambda)^2 - 4}|_{\lambda=\mu_n} = \sqrt[*]{\Delta(\mu_n)^2 - 4}.$$

As, by assumption, $\mu_n \notin \{\lambda_{2n}, \lambda_{2n-1}\}$, the root $\sqrt[*]{\Delta(\lambda)^2 - 4}$ is well-defined. In analogy with formula (46) define

$$\sqrt[*]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})} := \sqrt[*]{\Delta(\lambda)^2 - 4} \cdot \frac{\zeta_n(\lambda)}{\psi_n(\lambda)}, \qquad \lambda \in [\lambda_{2n-1}, \mu_n]. \tag{56}$$

As $\sqrt[*]{\Delta(\mu_n)^2 - 4} = y_1(1, \mu_n) - y_2'(1, \mu_n)$ is defined on all of W and analytic there, $\sqrt[*]{(\lambda_{2n} - \mu_n)(\mu_n - \lambda_{2n-1})}$ analytically extends to W as well and

$$z_n^\pm = \gamma_n \exp\left(\pm i \int_{\lambda_{2n-1}}^{\mu_n} \frac{\zeta_n(\lambda)}{\sqrt[*]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda \right). \tag{57}$$

To obtain the claimed estimates for $z_n^\pm$, we write $z_n^\pm$ as a product

$$z_n^+ = u_n^+ v_n^+ \quad \text{and} \quad z_n^- = u_n^- v_n^- \tag{58}$$

where

$$u_n^\pm = \gamma_n \exp\left(\pm i \int_{\lambda_{2n-1}}^{\mu_n} \frac{1}{\sqrt[*]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda \right) \tag{59}$$

and

$$v_n^\pm = \exp\left(\pm i \int_{\lambda_{2n-1}}^{\mu_n} \frac{\zeta_n(\lambda) - 1}{\sqrt[*]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda \right) \tag{60}$$

Arguing as in the proof of Lemma 9.1 in [10] one concludes that $u_n^\pm$ are analytic on $W \setminus Z_n$. Together with the analyticity of $z_n^\pm$ on $W \setminus Z_n$ it then follows that $v_n^\pm$ are analytic on $W \setminus Z_n$ as well. For $q \in W \setminus Z_n$ with $\mu_n = \lambda_{2n-1}$, one easily sees that

$$u_n^\pm = \gamma_n \quad \text{and} \quad v_n^\pm = 1. \tag{61}$$

A straightforward computation shows that for $q \in W \setminus Z_n$ with $\mu_n = \lambda_{2n}$

$$u_n^{\pm} = -\gamma_n \quad v_n^{\pm} = 1. \tag{62}$$

We will see that both, $u_n^{\pm}$ and $v_n^{\pm}$, continuously extend to all of W and that they admit asymptotics for $n \to \infty$ which allow to prove the asymptotics of $z_n^{\pm}$, claimed in Proposition 4.1. The quantities $u_n^{\pm}$ and $v_n^{\pm}$ will be studied separately. Let us begin with the $u_n^{\pm}$'s. To this end introduce for any q in W,

$$\Delta_n(\lambda) = 4\,\frac{\lambda - \lambda_0}{\pi_n^2}\left(\prod_{m \neq n} \frac{\lambda_{2m} - \lambda}{\pi_m^2}\right)\left(\prod_{m \neq n} \frac{\lambda_{2m-1} - \lambda}{\pi_m^2}\right). \tag{63}$$

Hence for λ in U_n, the principal branch of the square root $\sqrt[+]{\Delta_n(\lambda)}$ is well-defined. Furthermore recall that in section 4, we have introduced for any $n \geq 1$ and $q \in W$

$$\kappa_n(q) = \log{(-1)^n y_2'(1, \mu_n)} \tag{64}$$

where $\mu_n = \mu_n(q)$ is the n'th Dirichlet eigenvalue. Here log denotes the principal branch of the logarithm.

Proposition 4.2. *For any $n \geq 1$ and $q \in W \setminus Z_n$*

$$u_n^{\pm} = 2(\tau_n - \mu_n) \pm i\frac{2\pi n}{\sqrt[+]{\Delta_n(\mu_n)}} 2\sinh\kappa_n. \tag{65}$$

In particular, for any $n \geq 1$, $u_n^{\pm}$ extend analytically to all of W. For $q \in Z_n$,

$$u_n^{\pm} = 2(\tau_n - \mu_n) \pm 2i\sqrt[*]{-(\tau_n - \mu_n)^2}. \tag{66}$$

Remark 4.1. *In Appendix A, Proposition 4.2 is used to derive the formula for the differential of the Birkhoff map at $q = 0$ by a short calculation.*

Proof of Proposition 4.2. By the definition (64) of κ_n,

$$2\sinh\kappa_n = (-1)^n y_2'(1, \mu_n) - \left((-1)^n y_2'(1, \mu_n)\right)^{-1} = (-1)^{n-1}\sqrt[*]{\Delta(\mu_n)^2 - 4}.$$

Recall that by (21),

$$\Delta^2(\lambda) - 4 = \frac{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}{\pi_n^2}\Delta_n(\lambda). \tag{67}$$

By the definition (47) of $\zeta_n(\lambda)$ and the definition (41) of ψ_n one sees that for q real , $\zeta_n(\mu_n) > 0$ and $(-1)^{n-1}\psi_n(\mu_n) > 0$. Hence by the definition (56) of the root $\sqrt[*]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}$ it follows that for q real

$$(-1)^{n-1}\sqrt[*]{\Delta(\mu_n)^2 - 4} = \sqrt[+]{\Delta_n(\mu_n)}\,\frac{\sqrt[*]{(\lambda_{2n} - \mu_n)(\mu_n - \lambda_{2n-1})}}{\pi_n}.$$

As both sides of the last identity are analytic on W it holds for any $q \in W$. Hence it is to prove that

$$u_n^{\pm} = 2(\tau_n - \mu_n) \pm 2i \sqrt[*]{(\lambda_{2n} - \mu_n)(\mu_n - \lambda_{2n-1})}. \tag{68}$$

In the case where $\mu_n = \lambda_{2n-1}$, one obtains from (61) that the left and right hand side of (68) are equal to γ_n. If $\mu_n = \lambda_{2n}$, by (62), both sides of (68) equal $-\gamma_n$. It remains to verify (68) for $q \in W \setminus Z_n$ with $\mu_n \notin \{\lambda_{2n}, \lambda_{2n-1}\}$. Without loss of generality we may assume that $|\mu_n - \lambda_{2n-1}| \leq |\mu_n - \lambda_{2n}|$. Otherwise, we exchange the role of λ_{2n-1} and λ_{2n} and note that this will not change the value of $u_n^{\pm}$. Denote by $V_n \subseteq U_n$ a (small) open neighbourhood of $(\lambda_{2n-1}, \mu_n]$ which does not contain λ_{2n-1} nor λ_{2n}. There, the root $\sqrt[*]{(\lambda_{2n} - \mu)(\mu - \lambda_{2n-1})}$, defined on $[\lambda_{2n-1}, \mu_n]$, continuously extends to $V_n \cup \{\lambda_{2n-1}\}$. We again denote it by $\sqrt[*]{(\lambda_{2n} - \mu)(\mu - \lambda_{2n-1})}$. Furthermore, for $\mu \in V_n$, introduce

$$f_n^{\pm}(\mu) = \gamma_n \exp\left(\pm i \int_{\lambda_{2n-1}}^{\mu} \frac{d\lambda}{\sqrt[*]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} \right) \tag{69}$$

and

$$g_n^{\pm}(\mu) = 2(\tau_n - \mu) \pm 2i \sqrt[*]{(\lambda_{2n} - \mu)(\mu - \lambda_{2n-1})}. \tag{70}$$

Then

$$\lim_{\mu \to \lambda_{2n-1}} f_n^{\pm}(\mu) = \gamma_n = \lim_{\mu \to \lambda_{2n-1}} g_n^{\pm}(\mu),$$

$$\partial_\mu f_n^{\pm}(\mu) = \pm f_n^{\pm}(\mu) \frac{i}{\sqrt[*]{(\lambda_{2n} - \mu)(\mu - \lambda_{2n-1})}}$$

and

$$\partial_\mu g_n^{\pm}(\mu) = \pm g_n^{\pm}(\mu) \frac{i}{\sqrt[*]{(\lambda_{2n} - \mu)(\mu - \lambda_{2n-1})}}.$$

Hence f_n^{+} and g_n^{+} satisfy the same 1st order differential equation and have the same value at $\mu = \lambda_{2n-1}$. Hence

$$f_n^{+}(\mu) = g_n^{+}(\mu) \quad \forall \mu \in V_n. \tag{71}$$

In particular $f_n^{+}(\mu_n) = g_n^{+}(\mu_n)$. Similarly one has $f_n^{-}(\mu_n) = g_n^{-}(\mu_n)$ and the identity (68) is proved.

Note that the right hand side of (65) is defined on all of W. By the analyticity of $y_j(1, \lambda, q)$ $(j = 1, 2)$ on $\mathbb{C} \times W$, the analyticity of τ_n, μ_n, κ_n on W, and the analyticity of $\Delta_n(\lambda, q)$ on $U_n \times W$, it then follows that the right hand side of (65) is analytic on W. Formula (66) is obtained from (68). $\square$

As an application of Proposition 4.2 and the asymptotics of section 3 and section 4 we obtain

Corollary 4.1. *For q in W_N with $W_N \subseteq W \cap H^N_{0,\mathbb{C}}$ given as in Corollary 2.1,*

$$u_n^{\pm} = 2\langle q, \cos 2\pi nx\rangle \pm 2i\langle q, \sin 2\pi nx\rangle + \frac{1}{n^{N+1}}\ell_n^2$$

locally uniformly on W_N. On H_0^N, the error is uniformly bounded on bounded subsets of H_0^N.

Proof of Corollary 4.1. By Proposition 4.2, for $q \in W$,

$$u_n^{\pm} = 2(\tau_n - \mu_n) \pm i\frac{2\pi n}{\sqrt[+]{\Delta_n(\mu_n)}} 2\sinh\kappa_n.$$

By Theorem 2.5, for q in $W \cap H^N_{0,\mathbb{C}}$,

$$\kappa_n = \frac{1}{2\pi n}\left(\langle q, \sin 2\pi nx\rangle + \frac{1}{n^{N+1}}\ell_n^2\right)$$

and the Taylor expansion of $\sinh z$ at $z = 0$ then yields

$$\sinh\kappa_n = \frac{1}{2\pi n}\left(\langle q, \sin 2\pi nx\rangle + \frac{1}{n^{N+1}}\ell_n^2\right).$$

According to [12], Corollary 7.1, $\Delta_n(\lambda) = 1 + \frac{1}{n}\ell_n^2$ and hence

$$\left(\sqrt[+]{\Delta_n(\lambda)}\right)^{-1} = 1 + \frac{1}{n}\ell_n^2$$

uniformly for $\lambda \in U_n$. Furthermore, by Corollary 2.1, for $q \in W_N$,

$$2(\tau_n - \mu_n) = 2\langle q, \cos 2\pi nx\rangle + \frac{1}{n^{N+1}}\ell_n^2.$$

Altogether we get for q in W_N

$$\begin{aligned} u_n^{\pm} =& 2\langle q, \cos 2\pi nx\rangle + \frac{1}{n^{N+1}}\ell_n^2 \pm 2i\left(1 + \frac{1}{n}\ell_n^2\right)\left(\langle q, \sin 2\pi nx\rangle + \frac{1}{n^{N+1}}\ell_n^2\right) \\ =& 2\langle q, \cos 2\pi nx\rangle \pm 2i\langle q, \sin 2\pi nx\rangle + \frac{1}{n^{N+1}}\ell_n^2 \end{aligned}$$

as claimed. Going through the arguments of the proof one sees that the estimate holds locally uniformly on W_N and uniformly on bounded subsets of H_0^N. □

It remains to analyze the asymptotics for $v_n^{\pm}$. We need the following auxiliary result.

Lemma 4.1. *For q in $W \setminus Z_n$ with $|\mu_n - \lambda_{2n-1}| \leq |\mu_n - \lambda_{2n}|$*

$$\left|\int_{\lambda_{2n-1}}^{\mu_n} \frac{\zeta_n(\lambda) - \zeta_n(\lambda_{2n-1})}{\sqrt[*]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda\right| \leq (|\mu_n - \tau_n| + |\gamma_n|) \sup_{\lambda \in [\lambda_{2n-1}, \mu_n]} |\dot{\zeta}_n(\lambda)| \tag{72}$$

and

$$\left|\int_{\lambda_{2n-1}}^{\mu_n} \frac{\zeta_n(\lambda_{2n-1})-1}{\sqrt[*]{(\lambda_{2n}-\lambda)(\lambda-\lambda_{2n-1})}}d\lambda\right| \le 3\left(|\mu_n-\tau_n|+|\gamma_n|\right)\sup_{\lambda\in G_n}\frac{|\zeta_n(\lambda)-1|}{|\gamma_n|} \tag{73}$$

If $|\mu_n-\lambda_{2n-1}|\ge|\mu_n-\lambda_{2n}|$, (72) *and* (73) *hold if the roles of* λ_{2n-1} *and* λ_{2n} *are interchanged.*

Proof of Lemma 4.1. First, assume that $q\in W\setminus Z_n$ and

$$|\mu_n-\lambda_{2n-1}|\le|\mu_n-\lambda_{2n}|. \tag{74}$$

This implies that $\mu_n\neq\lambda_{2n}$. If $\mu_n=\lambda_{2n-1}$ then (72) and (73) hold as their left-hand sides vanish. Now assume that $\mu_n\neq\lambda_{2n-1}$. By the mean value theorem

$$\zeta_n(\lambda)=f_n(\lambda)(\lambda-\lambda_{2n-1})+\zeta_n(\lambda_{2n-1}) \tag{75}$$

where for λ in U_n

$$f_n(\lambda)=\int_0^1\dot{\zeta}_n\left(\lambda_{2n-1}+s(\lambda-\lambda_{2n-1})\right)ds.$$

Hence

$$\sup_{\lambda\in[\lambda_{2n-1},\mu_n]}|f_n(\lambda)|\le\sup_{\lambda\in[\lambda_{2n-1},\mu_n]}\left|\dot{\zeta}_n(\lambda)\right|. \tag{76}$$

(Here we assume (without loss of generality) that U_n is convex.) It follows from (75) that, with $\lambda(t)=\lambda_{2n-1}+t(\mu_n-\lambda_{2n-1})$,

$$\left|\int_{\lambda_{2n-1}}^{\mu_n}\frac{\zeta_n(\lambda)-\zeta_n(\lambda_{2n-1})}{\sqrt[*]{(\lambda_{2n}-\lambda)(\lambda-\lambda_{2n-1})}}d\lambda\right|=\left|\int_0^1 f_n(\lambda)\frac{\sqrt{\lambda-\lambda_{2n-1}}}{\sqrt{\lambda_{2n}-\lambda}}(\mu_n-\lambda_{2n-1})dt\right|.$$

In view of (74), $|\lambda(t)-\lambda_{2n-1}|\le|\lambda(t)-\lambda_{2n}|$ for any $0\le t\le 1$ and thus

$$\begin{aligned}&\left|\int_0^1\frac{f_n(\lambda)\sqrt{\lambda-\lambda_{2n-1}}}{\sqrt{\lambda_{2n}-\lambda}}(\mu_n-\lambda_{2n-1})dt\right|\\ \le&|\mu_n-\lambda_{2n-1}|\sup_{\lambda\in[\lambda_{2n-1},\mu_n]}|f_n(\lambda)|.\end{aligned}$$

As $|\mu_n-\lambda_{2n-1}|\le|\mu_n-\tau_n|+|\gamma_n|/2$, the claimed estimate (72) then follows from (76). Next consider the term

$$(\zeta_n(\lambda_{2n-1})-1)\int_{\lambda_{2n-1}}^{\mu_n}\frac{d\lambda}{\sqrt[*]{(\lambda_{2n}-\lambda)(\lambda-\lambda_{2n-1})}}.$$

To estimate the integral, let $\lambda(t) = \lambda_{2n-1} + t(\mu_n - \lambda_{2n-1})$ to get

$$\left| \int_{\lambda_{2n-1}}^{\mu_n} \frac{d\lambda}{\sqrt[*]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} \right| = \left| \int_0^1 \frac{\sqrt{\mu_n - \lambda_{2n-1}}}{\sqrt{\lambda_{2n} - \lambda}} \frac{dt}{\sqrt{t}} \right|$$

$$\le \sqrt{2} \frac{2|\mu_n - \lambda_{2n-1}|^{\frac{1}{2}}}{|\gamma_n|^{\frac{1}{2}}} \tag{77}$$

where we used again that by (74), $|\lambda_{2n} - \lambda(t)| \ge |\gamma_n/2|$ for $0 \le t \le 1$. In view of Lemma 3.2 (ii) and as

$$2|\mu_n - \lambda_{2n-1}|^{\frac{1}{2}} |\gamma_n|^{\frac{1}{2}} \le |\mu_n - \lambda_{2n-1}| + |\gamma_n| \le |\mu_n - \tau_n| + \frac{3}{2}|\gamma_n|$$

the claimed estimate (73) follows. The case when $|\mu_n - \lambda_{2n-1}| \ge |\mu_n - \lambda_{2n}|$ is treated in a similar way. □

Corollary 4.2. *For any $n \ge 1$, v_n^+ and v_n^- continuously extend to all of W. These extensions are again denoted by $v_n^\pm$. For $q \in Z_n$ with $\mu_n \ne \tau_n$*

$$v_n^\pm = \exp\left(\pm i \int_{\tau_n}^{\mu_n} \frac{\zeta_n(\lambda) - \zeta_n(\tau_n)}{\sqrt[*]{-(\tau_n - \lambda)^2}} d\lambda \right) \tag{78}$$

whereas for $q \in Z_n$ with $\mu_n = \tau_n$, $v_n^\pm = 1$.

Proof of Corollary 4.2. Let $q \in Z_n$ with $\mu_n \ne \tau_n$. It follows from (25), (46), and (51) that for $q \in W \setminus Z_n$,

$$\int_{\lambda_{2n-1}}^{\lambda_{2n}} \frac{\zeta_n(\lambda) - 1}{\sqrt{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda = 0 \tag{79}$$

for any choice of the sign of the root. In particular, (79) holds for the $*$-root and thus

$$\int_{\lambda_{2n-1}}^{\mu_n} \frac{\zeta_n(\lambda) - 1}{\sqrt[*]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda = \int_{\lambda_{2n}}^{\mu_n} \frac{\zeta_n(\lambda) - 1}{\sqrt[*]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda.$$

Hence without loss of generality we may assume that $|\mu_n - \lambda_{2n-1}| \le |\mu_n - \lambda_{2n}|$ for q, as otherwise we simply interchange the role of λ_{2n} and λ_{2n-1} in (60). Moreover, we may assume that the isolating neighbourhood U_n of q is also an isolating neighbourhood of p for any p in some neighbourhood W_q of q. To compute the limit $\lim_{p \to q,\, p \in W_q \setminus Z_n} v_n^\pm$ split

$$\int_{\lambda_{2n-1}}^{\mu_n} \frac{\zeta_n(\lambda) - 1}{\sqrt[*]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda$$

into two parts by writing

$$\zeta_n(\lambda) - 1 = (\zeta_n(\lambda) - \zeta_n(\lambda_{2n-1})) + (\zeta_n(\lambda_{2n-1}) - 1).$$

Then

$$\lim_{p\to q,\, p\in W_q\setminus Z_n} \int_{\lambda_{2n-1}}^{\mu_n} \frac{\zeta_n(\lambda)-\zeta_n(\lambda_{2n-1})}{\sqrt[*]{(\lambda_{2n}-\lambda)(\lambda-\lambda_{2n-1})}} d\lambda = \int_{\tau_n}^{\mu_n} \frac{\zeta_n(\lambda)-\zeta_n(\tau_n)}{\sqrt[*]{-(\tau_n-\lambda)^2}} d\lambda \Big|_{p=q}$$

and in view of Lemma 3.2 and (77),

$$\lim_{p\to q,\, p\in W_q\setminus Z_n\, |\mu_n-\lambda_{2n-1}|\le|\lambda_{2n}-\mu_n|} \int_{\lambda_{2n-1}}^{\mu_n} \frac{\zeta_n(\lambda_{2n-1})-1}{\sqrt[*]{(\lambda_{2n}-\lambda)(\lambda-\lambda_{2n-1})}} d\lambda = 0.$$

By the same reason the integral

$$\int_{\lambda_{2n}}^{\mu_n} \frac{\zeta_n(\lambda_{2n})-1}{\sqrt[*]{(\lambda_{2n}-\lambda)(\lambda-\lambda_{2n-1})}} d\lambda$$

converges to zero as $p\to q$ for $p\in W_q$ with $|\mu_n-\lambda_{2n-1}|\ge|\lambda_{2n}-\mu_n|$. Altogether we thus have shown that

$$\lim_{p\to q,\, p\in W_q\setminus Z_n} v_n^{\pm}(p) = \exp\left(\pm i \int_{\tau_n}^{\mu_n} \frac{\zeta_n(\lambda)-\zeta_n(\tau_n)}{\sqrt[*]{-(\tau_n-\lambda)^2}} d\lambda\right)$$

as claimed. Now let us consider the case where $q\in Z_n$ with $\mu_n=\tau_n$. From (78) one concludes that

$$\lim_{p\to q,\, p\in W_q\cap Z_n,\, \mu_n\ne\tau_n} v_n^{\pm}(p) = 1$$

and from Lemma 4.1 and Lemma 3.2 one sees that

$$\lim_{p\to q,\, p\in W_q\setminus Z_n,\, \mu_n\ne\tau_n} v_n^{\pm}(p) = 1.$$

□

It remains to study the asymptotics of $v_n^{\pm}$, now defined on all of W.

Corollary 4.3. *For q in W*

$$v_n^{\pm} = 1 + \frac{1}{n}\ell_n^2 \tag{80}$$

locally uniformly on W. On L_0^2, (80) *holds uniformly on bounded subsets of L_0^2.*

Proof of Corollary 4.3. We want to apply Lemma 4.1. First note that in view of Corollary 4.2, the estimates of Lemma 4.1 hold on all of W, not only on $W\setminus Z_n$. Furthermore, by shrinking the isolating neighbourhoods we get from Lemma 3.2 and Cauchy's estimate

$$\sup_{\lambda\in U_n} |\dot{\zeta}_n(\lambda)| = \frac{1}{n}\ell_n^2.$$

By Lemma 3.2,

$$\sup_{\lambda\in G_n} \frac{|\zeta_n(\lambda)-1|}{|\gamma_n|} = \frac{1}{n}\ell_n^2.$$

Using that $|e^x-1|\le|x|e^{|x|}$, the claimed estimate then follows indeed from Lemma 4.1. Going through the arguments of the proof one sees that (80) holds locally uniformly on W and uniformly on bounded subsets of L_0^2. □

Proof of Proposition 4.1. By Corollary 4.2 and Lemma 4.1, $u_n^{\pm}$ and $v_n^{\pm}$ are defined and continuous on W for any $n \geq 1$. Hence the identities (58) extend to all of W,

$$z_n^+ = u_n^+ v_n^+ \quad \text{and} \quad z_n^- = u_n^- v_n^-. \tag{81}$$

By Corollary 4.1 and Corollary 4.3, it follows that for q in W_N,

$$\begin{aligned} z_n^{\pm} &= \left(2\langle q, \cos 2\pi nx\rangle \pm 2i\langle q, \sin 2\pi nx\rangle + \frac{1}{n^{N+1}}\ell_n^2\right)\left(1 + \frac{1}{n}\ell_n^2\right) \\ &= 2\hat{q}_{\mp n} + \frac{1}{n^{N+1}}\ell_n^2. \end{aligned}$$

Going through the arguments of the proof one sees that the estimate holds locally uniformly on W_N and uniformly on bounded subsets of H_0^N. This proves Proposition 4.1. ☐

5 Proof of Theorem 1.4

In this section we prove the asymptotics of the Birkhoff map claimed in Theorem 1.4 and use them to derive further properties of this map.

Proof of Theorem 1.4. For any $N \in \mathbb{Z}_{\geq 0}$, let $W_N \subseteq W \cap H_{0,\mathbb{C}}^N$ be the neighbourhood of H_0^N given by Theorem 2.4. For any $q \in W$, $\Phi(q)$ is given by $\Phi(q) = (z_n(q))_{n \neq 0}$, where for any $n \geq 1$,

$$z_{-n} = x_n + iy_n = \xi_n \frac{z_n^+}{2} e^{i\beta_n}, \quad \text{and} \quad z_n = x_n - iy_n = \xi_n \frac{z_n^-}{2} e^{-i\beta_n}. \tag{82}$$

By Proposition 3.1 and Lemma 3.1

$$\xi_n = \frac{1}{\sqrt{\pi n}}\left(1 + \frac{1}{n}\ell_n^2\right) \quad \text{and} \quad e^{i\beta_n} = 1 + O\left(\frac{1}{n}\right) \tag{83}$$

whereas by Proposition 4.1, for $q \in W_N$,

$$\frac{z_n^+}{2} = \hat{q}_{-n} + \frac{1}{n^{N+1}}\ell_n^2. \tag{84}$$

Hence

$$z_{-n} = \frac{1}{\sqrt{n\pi}}\left(\hat{q}_{-n} + \frac{1}{n^{N+1}}\ell_n^2\right).$$

A similar estimate holds for z_n. Note that all the asymptotic estimates referred to hold locally uniformly on W_N. As by Theorem 1.3, $z_{\pm n}$ are analytic on W for any $n \geq 1$ it follows from Theorem A.5 in [10] that $\Phi - \Phi_0 : W_N \to \mathfrak{h}_{\mathbb{C}}^{N+3/2}$ is analytic. This proves the first part of Theorem 1.4.

Going through the above arguments and using that by Proposition 3.1, Lemma 3.1, and Proposition 4.1, the estimates (83) and (84) hold uniformly on bounded sets of H_0^N, it follows that $A := (\Phi - \Phi_0) : H_0^N \to \mathfrak{h}^{N+3/2}$ is bounded. ☐

Proposition 5.1. *Let $N \in \mathbb{Z}_{\geq 0}$. The restriction of Φ^{-1} to $\mathfrak{h}^{N+1/2}$ is of the form $\Phi^{-1} = \Phi_0^{-1} + B$, with $B = -\Phi_0^{-1} \circ A \circ \Phi^{-1}$. B is a map from $\mathfrak{h}^{N+1/2}$ to H_0^{N+1}. It is bounded and real analytic.*

Proof. First let us verify the formula for B. Clearly, it follows from

$$Id_{\mathfrak{h}^{N+1/2}} = (\Phi_0 + A) \circ \Phi^{-1} = Id_{\mathfrak{h}^{N+1/2}} + \Phi_0 \circ B + A \circ \Phi^{-1}$$

that

$$B = -\Phi_0^{-1} \circ A \circ \Phi^{-1}.$$

As B is given by the composition of real analytic maps,

$$\mathfrak{h}^{N+1/2} \xrightarrow{\Phi^{-1}} H_0^N \xrightarrow{A} \mathfrak{h}^{N+3/2} \xrightarrow{-\Phi_0^{-1}} H_0^{N+1},$$

it is itself real analytic. It remains to prove that for any $N \in \mathbb{Z}_{\geq 0}$, $B : \mathfrak{h}^{N+1/2} \to H^{N+1}$ is bounded. First note that for any $N \in \mathbb{Z}_{\geq 0}$, the inverse of the weighted Fourier transform $\Phi_0^{-1} : \mathfrak{h}^{N+1/2} \to H_0^N$ and, by Theorem 1.4, the nonlinear map $A : H_0^N \to \mathfrak{h}^{N+3/2}$ are bounded for any $N \in \mathbb{Z}_{\geq 0}$. Furthermore, the boundedness of $\Phi^{-1} : \mathfrak{h}^{N+1/2} \to H_0^N$ follows, in the case $N = 0$, from the identity $\sum_{n \geq 1} 2\pi n I_n = \frac{1}{2} \|q\|_{L^2}$, established in [10], Theorem E.1, and, in the case $N \in \mathbb{Z}_{\geq 1}$, from [15], Theorem 2.4 and Theorem 2.6. More precisely, in [15] it is shown that for any $q \in H^N$ with $\lambda_0(q) = 0$ and $N \in \mathbb{Z}_{\geq 1}$, $\|\partial_x^N q\|_{L^2}$ can be bounded in terms of $\|J_n(q)\|_{\mathfrak{h}^{N+1/2}}$. Note that for any $p \in H_0^N$, $q = p - \lambda_0(p)$ satisfies $\lambda_0(q) = 0$ and hence, as $N \geq 1$, $\|\partial_x^N q\|_{L^2} = \|\partial_x^N p\|_{L^2}$. Furthermore, the quantities $(J_n^2(q))_{n \geq 1}$, introduced in [15], formula (1.2), can be shown to coincide with the action variables $(I_n(p))_{n \geq 1}$, introduced in [10] (cf formula (7.2)). Combining these results it follows that $B = -\Phi_0^{-1} \circ A \circ \Phi^{-1} : \mathfrak{h}^{N+1/2} \to H_0^{N+1}$ is bounded for any $N \in \mathbb{Z}_{\geq 0}$. $\square$

By Cauchy's estimate, Theorem 1.4 yields the following asymptotics of the differential of Φ.

Corollary 5.1. *For any $q \in W_N$ with $N \geq 0$, the differential of the Birkhoff map*

$$d_q \Phi : H_{0,\mathbb{C}}^N \to \mathfrak{h}_{\mathbb{C}}^{N+1/2}$$

satisfies the asymptotic estimate

$$d_q \Phi(f) = \left(\frac{1}{\sqrt{n\pi}} \hat{f}_n + \mathcal{R}_n(f) \right)_{n \neq 0}$$

where

$$(\mathcal{R}_n(f))_{n \neq 0} \in \mathfrak{h}^{N+3/2}.$$

As an immediate application of Corollary 5.1 we obtain

Corollary 5.2. *For any $q \in W_N$ with $N \geq 0$,*

$$d_q\Phi : H^N_{0,\mathbb{C}} \to \mathfrak{h}^{N+1/2}_{\mathbb{C}}$$

is a compact perturbation of the (weighted) Fourier transform. In particular, it satisfies the Fredholm alternative.

Corollary 5.2 is already proved in [10]. The proof argues by approximation and uses quite complicated computations. It is based on the fact that the set of finite gap potentials is dense in H^N_0. In the set-up presented here, its proof is straightforward given the asymptotic estimates stated in section 2.

6 Proof of Theorem 1.1 and Theorem 1.2

In this section we prove Theorem 1.1 and Theorem 1.2. First we need to derive some auxilary results. Let $N \in \mathbb{Z}_{\geq 0}$ and $c \in \mathbb{R}$ be given. For any q in H^N_c, write $q = p + c$ where $p \in H^N_0$ and $\int_0^1 q(x)dx = \hat{q}_0 = c$. Then

$$\mathcal{H}(p + c) = \mathcal{H}(p) + 3c \int_0^1 p^2 dx + c^3.$$

Note that $\frac{1}{2}\int_0^1 p^2 dx$ is the second Hamiltonian in the KdV hierarchy. By Parseval's identity for KdV (cf [10], Appendix E)

$$\frac{1}{2}\int_0^1 p^2 dx = \sum_{n \geq 1} 2\pi n I_n$$

and thus the KdV frequencies satisfy

$$\omega_n(q) = \omega_n(p) + 12cn\pi \quad \forall n \geq 1, \tag{85}$$

and the KdV flow is given by

$$u(t) \equiv S^t(q) = S^t_c(p) + c \tag{86}$$

where S^t_c denotes the flow on H^N_0 corresponding to the Hamiltonian

$$\mathcal{H}_c(p) := \mathcal{H}(p) + 3c \int_0^1 p^2 dx.$$

The equations of motion corresponding to $\mathcal{H}_c$ read, when expressed in Birkhoff coordinates $(z_n)_{n \neq 0}$,

$$\dot{z}_n = i\omega^c_n z_n \quad \text{and} \quad \dot{z}_{-n} = -i\omega^c_n z_{-n}$$

where

$$\omega^c_n \equiv \omega^c_n(p) = \omega_n(p) + 12cn\pi. \tag{87}$$

Define $\omega_{-n}^c := -\omega_n^c$ $(n \geq 1)$. Then

$$v(t) \equiv \Omega_c^t(z) = \left(e^{i\omega_n^c t} z_n\right)_{n \neq 0} \tag{88}$$

is the flow of $\mathcal{H}_c$ expressed in (complex) Birkhoff coordinates. Here, by a slight abuse of terminology, ω_n^c is viewed as a function of z. Note that Φ conjugates the flow maps S_c^t and Ω_c^t,

$$S_c^t = \Phi^{-1} \circ \Omega_c^t \circ \Phi.$$

With $B := \Phi^{-1} - \Phi_0^{-1}$, we get

$$S_c^t = \Phi_0^{-1} \circ \Omega_c^t \circ \Phi + B \circ \Omega_c^t \circ \Phi. \tag{89}$$

We now analyse the map $\Phi_0^{-1} \circ \Omega_c^t \circ \Phi$ in more detail. First note that for any $z = (z_n)_{n \neq 0} \in \mathfrak{h}^{N+1/2}$,

$$\Phi_0^{-1}(z)(x) = \sum_{n \neq 0} \sqrt{|n|\pi} z_n e^{2\pi i n x}.$$

Hence for any $p \in H_0^N$ and $t, c \in \mathbb{R}$,

$$\Phi_0^{-1} \circ \Omega_c^t \circ \Phi(p) = \sum_{n \neq 0} e^{i\omega_n^c t} \sqrt{|n|\pi} z_n(p) e^{2\pi i n x} \tag{90}$$

where $(z_n(p))_{n \neq 0} = \Phi(p)$. For the proof of item (i) of Theorem 1.1 we need to consider the KdV flow on all of H^N. For any $t \in \mathbb{R}$, introduce

$$E^t : H^N \times \mathfrak{h}^{N+1/2} \to \mathfrak{h}^{N+1/2}, \ (q, z) \mapsto (e^{i\omega_n(q)t} z_n)_{n \neq 0}.$$

Denoting by Π the projection $\Pi : H^N \to H_0^N$, $q \mapsto q - \hat{q}_0$ it follows that for any $q \in H^N$,

$$E^t(q, \Phi \circ \Pi(q)) = (e^{i\omega_n(q)t} z_n(\Pi(q)))_{n \neq 0} = \Omega_{\hat{q}_0}^t \circ \Phi(\Pi(q)). \tag{91}$$

Lemma 6.1. *For any given $N \in \mathbb{Z}_{\geq 0}$ and $t \in \mathbb{R}$, the map $E^t : H^N \times \mathfrak{h}^{N+1/2} \to \mathfrak{h}^{N+1/2}$ is continuous.*

Proof. For any $q, p \in H^N$, $z, w \in \mathfrak{h}^{N+1/2}$, and for any $K > 0$ one has

$$\|E^t(q, z) - E^t(p, w)\|_{N+1/2}^2 = \sum_{n \neq 0} |n|^{2N+1} |e^{i\omega_n(q)t} z_n - e^{i\omega_n(p)t} w_n|^2.$$

As the KdV frequencies are real valued functions on H^N

$$\begin{aligned} |e^{i\omega_n(q)t} z_n - e^{i\omega_n(p)t} w_n|^2 =& |z_n - e^{i(\omega_n(p) - \omega_n(q))t} w_n|^2 \\ \leq& 2|z_n - w_n|^2 + 2|1 - e^{i(\omega_n(p) - \omega_n(q))t}|^2 |w_n|^2 \end{aligned}$$

and hence

$$\|E^t(q,z) - E^t(p,w)\|^2_{N+1/2} \leq 2\|z-w\|^2_{N+1/2} + 8\sum_{|n|\geq K} |n|^{2K+1}|w_n|^2$$
$$+2\sum_{0<|n|<K} |n|^{2N+1}|e^{i(\omega_n(p)-\omega_n(q))t} - 1|^2 |w_n|^2. \quad (92)$$

By (85),

$$\omega_n(p) - \omega_n(q) = 12n\pi(\widehat{p-q})_0 + O(1)$$

locally uniformly on $H^N \times H^N$. Moreover, it follows from (85) and [14], Theorem 1.9 that for any $n \neq 0$, the n'th frequency $\omega_n : H^N \to \mathbb{R}$ is continuous. This combined with (92) implies the statement of the lemma. □

Proof of Theorem 1.1. Let $N \in \mathbb{Z}_{\geq 0}$. For any $q \in H^N$ let $p = \Pi(q) = q - \hat{q}_0$. Then by (86) and (89), with $c = \hat{q}_0$,

$$S^t(q) = S^t_c(p) + c = \Phi_0^{-1} \circ \Omega^t_c \circ \Phi(p) + B \circ \Omega^t_c \circ \Phi(p) + c.$$

Substituting $\Phi = \Phi_0 + A$ into $\Phi_0^{-1} \circ \Omega^t_c \circ \Phi(p)$ then yields

$$\Phi_0^{-1} \circ \Omega^t_c \circ \Phi(p) = \sum_{n\neq 0} e^{i\omega^c_n t}\hat{p}_n e^{2\pi i n x} + \Phi_0^{-1} \circ \Omega^t_c \circ A(p)$$

where by a slight abuse of terminology we denote by $\Omega^t_c \circ A(p)$ the element $\Omega^t_c \circ A(p) = \left(e^{i\omega^c_n(p)t}a_n(p)\right)_{n\neq 0}$ and $A(p) = (a_n(p))_{n\neq 0}$. Taking into account that for any $n \neq 0$, $\hat{q}_n = \hat{p}_n$ and $\omega^c_n(p) = \omega_n(q)$ we conclude

$$\begin{aligned} R^t(q) =& S^t(q) - \sum_n e^{i\omega_n t}\hat{q}_n e^{2\pi i n x} \\ =& B \circ \Omega^t_c \circ \Phi(p) + \Phi_0^{-1} \circ \Omega^t_c \circ A(p). \end{aligned} \quad (93)$$

Statements (ii) and (iii) of Theorem 1.1 then follow from Theorem 1.4, Proposition 5.1, and the boundedness of $\Phi = \Phi_0 + A$. To prove item (i) note that by (91) and (93),

$$R^t(q) = (B \circ E^t)(q, \Phi \circ \Pi(q)) + (\Phi_0^{-1} \circ E^t)(q, A \circ \Pi(q)). \quad (94)$$

It then follows from Lemma 6.1, Theorem 1.4, and Proposition 5.1 that R^t is continuous.

To prove statement (iv) write for $n \in \mathbb{Z}$ arbitrary,

$$\hat{u}_n(t) = e^{i\omega_n t}(\hat{q}_n + \tilde{\rho}_n(t)),$$

where $\omega_0 := 0$ and

$$\tilde{\rho}_n(t) := \widehat{R}^t_n(q)e^{-i\omega_n t}.$$

By the definition of R^t, $\hat{R}^t_0(q) = 0$ implying that $\tilde{\rho}_0(t)$ vanishes identically. Moreover $\tilde{\rho}_n(0) = 0$ for any $n \in \mathbb{Z}$ as $R^0(q) = 0$. Clearly $\partial_t \hat{u}_n(t) = i\omega_n \hat{u}_n(t) + e^{i\omega_n t}\partial_t \tilde{\rho}_n(t)$ and when substituted into the KdV equation

$$-\partial_t \hat{u}_n(t) + 8i\pi^3 n^3 \hat{u}_n(t) + 6i\pi n \sum_l \hat{u}_l(t)\hat{u}_{n-l}(t) = 0,$$

one gets

$$-i\omega_n \hat{u}_n(t) - e^{i\omega_n t}\partial_t \tilde{\rho}_n(t) + 8i\pi^3 n^3 \hat{u}_n(t) + 6i\pi n \sum_l \hat{u}_l(t)\hat{u}_{n-l}(t) = 0$$

or

$$ie^{i\omega_n t}\partial_t \tilde{\rho}_n(t) = (\omega_n - 8\pi^3 n^3)\hat{u}_n(t) - 6\pi n \sum_l \hat{u}_l(t)\hat{u}_{n-l}(t). \tag{95}$$

Recall from [10], p 229

$$\omega_n(q) = 8\pi n(\tau_n + \lambda_0/2 - \sum_{m \geq 1}(\sigma^n_m - \tau_m)), \tag{96}$$

where λ_0, τ_n, and σ^n_m have been introduced either in Section 1 or Section 2. For $N \geq 1$, $H^N \hookrightarrow L^2$ is compact. As $L^2 \to \mathbb{R}, q \mapsto \lambda_0(q)$ is continuous, it then follows that $H^N \to \mathbb{R} : q \mapsto \lambda_0(q)$ is compact. By Theorem 2.4

$$\tau_n = \hat{q}_0 + n^2\pi^2 + \frac{1}{n}\ell^2_n$$

whereas by Proposition 2.2 and Theorem 2.3 , uniformly in n

$$\sigma^n_m - \tau_m = \frac{\gamma^2_m}{m}\ell^2_m = \frac{1}{m^{2N+1}}\ell^1_m.$$

Both asymptotic estimates hold uniformly on bounded subsets of H^N. Hence formula (96) leads to the asymptotics

$$\omega_n(q) = 8\pi^3 n^3 + O(n) \tag{97}$$

uniformly on bounded subsets of H^N and statement (iv) follows. □

Remark 6.1. *As pointed out in Remark 1.1, the restrictions of R^t and $\partial_t R^t$ to H^N_c are real analytic. To formulate this result more precisely, for any $c \in \mathbb{R}$, denote by $H^N_{c,\mathbb{C}}$ the complexification of H^N_c,*

$$H^N_{c,\mathbb{C}} := \{q \in H^N_{\mathbb{C}} \,|\, \int_0^1 q(x)dx = c\} = c + H^N_{0,\mathbb{C}}.$$

Note that $H^N_{c,\mathbb{C}}$ is not *an open complex neighbourhood of H^N_c in $H^N_{\mathbb{C}}$ as for q in $H^N_{c,\mathbb{C}}$ the value of $\hat{q}_0$ is kept fixed. Let E be a real Banach space and denote*

by $E_{\mathbb{C}}$ its complexification. A map $f : H^N_c \to E$ is said to be real analytic if f extends to an analytic map $f : W \to E_{\mathbb{C}}$ where W is an open neigbourhood of H_c in $H^N_{c,\mathbb{C}}$. In view of the fact mentioned above that $\hat{R}^t_0 = 0$ for any $t \in \mathbb{R}$, R^t maps H^N into H^{N+1}_0 and $\partial_t R^t$ maps H^N into H^{N-1}_0 . Theorem 1.1 can then be amended as follows:

(v) for any $N \in \mathbb{Z}_{\geq 0}$ and $c \in \mathbb{R}$, $R^t_{|H^N_c} : H^N_c \to H^{N+1}_0$ is real analytic;

(vi) for any $N \in \mathbb{Z}_{\geq 1}$ and $c \in \mathbb{R}$, $\partial_t R^t_{|H^N_c} : H^N_c \to H^{N-1}_0$ is real analytic.

To see that $R^t_{|H^N_c}$ is real analytic, note that it follows from [1], Theorem 1 and formula (85) *that for any $q \in H^N_c$,*

$$\omega_n(q) = 8\pi^3 n^3 + 12cn\pi + O(1), \tag{98}$$

locally uniformly in a complex neighbourhood V of H^N_c in $H^N_{c,\mathbb{C}}$ and that for any $n \geq 1$, ω_n is real analytic on V. One then concludes that for any $t \in \mathbb{R}$

$$E^t : V \times \mathfrak{h}^{N+1/2}_{\mathbb{C}} \to \mathfrak{h}^{N+1/2}_{\mathbb{C}},\ (q, (z_n)_{n\neq 0}) \mapsto (e^{i\omega_n(q)t} z_n)_{n\neq 0}$$

is analytic. Together with the analyticity of Φ as well as Φ^{-1} and hence of $B = \Phi^{-1} - \Phi^{-1}_0$, it then follows from (94) *that $R^t_{|H^N_c}$ is real analytic. The analyticity of $\partial_t R^t_{|H^N_c}$ stated in item (vi) is proved in a similar fashion.*

We now turn to the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $\epsilon, M > 0$ be given and let $q \in H^N$ satisfy $\|q\|_{H^N} \leq M$. Apply $(Id - P_L)$ to $S^t(q) = \sum_{n\in\mathbb{Z}} e^{i\omega_n t}\hat{q}_n e^{2\pi inx} + R^t(q)$,

$$(Id - P_L)S^t(q) = \sum_{|n|>L} e^{i\omega_n t}\hat{q}_n e^{2\pi inx} + (Id - P_L)R^t(q).$$

Note that

$$\|\sum_{|n|>L} e^{i\omega_n t}\hat{q}_n e^{2\pi inx}\|_{H^N} = \|(Id - P_L)q\|_{H^N}. \tag{99}$$

By Theorem 1.4 (iii), $B_M := \{R^t(p) \,|\, t \in \mathbb{R}, p \in H^N, \|p\|_{H^N} \leq M\}$ is bounded in H^{N+1} and thus by the Sobolev embedding theorem relatively compact in H^N. Hence there exists $L_\star \in \mathbb{N}$ such that for any $L \geq L_\star$

$$\|(Id - P_L)r\|_{H^N} < \epsilon \quad \forall r \in B_M.$$

Thus

$$\begin{aligned}\|(Id - P_L)S^t(q)\|_{H^N} \leq& \|(Id - P_L)q\|_{H^N} + \|(Id - P_L)R^t(q)\|_{H^N}\\ \leq& \|(Id - P_L)q\|_{H^N} + \epsilon\end{aligned}$$

and

$$\|(Id - P_L)S^t(q)\|_{H^N} \geq \|(Id - P_L)q\|_{H^N} - \epsilon.$$

□

There are other ways of approximating the KdV flow than the one considered in Theorem 1.2. As an alternative to the projection of the solution of KdV onto the space of trigonometric polynomials of order L one could involve the orthogonal projection $Q_L : \mathfrak{h}^{1/2} \to \mathfrak{h}^{1/2}$ onto the $2L$ dimensional $\mathbb{R}$-vector space

$$\{(z_k)_{k\neq 0} |\ z_k = 0\ \forall |k| > L\}$$

and study

$$\Phi^{-1} \circ Q_L \circ \Phi \circ S_c^t(p) = \Phi^{-1} \circ Q_L \circ \Omega_c^t \circ \Phi(p).$$

Results similar to the ones of Theorem 1.2 can be obtained for such a type of approximation.

7 Appendix A

In this appendix we use Proposition 4.2 to derive the formula for the differential of the Birkhoff map at $q = 0$ by a short calculation – see [10] for an alternative, but lengthier derivation. Recall from (81) and (82) that

$$z_{-n} = x_n + iy_n = \xi_n e^{i\beta_n} \frac{u_n^+}{2} v_n^+.$$

By Proposition 4.2

$$\frac{u_n^+}{2} = (\tau_n - \mu_n) + i \frac{2\pi n}{\sqrt[+]{\Delta_n(\mu_n)}} \sinh \kappa_n.$$

For $q = 0$, $\tau_n - \mu_n = 0$, $\kappa_n = 0$, $v_n^+ = 1$ (Corollary 4.2), $\xi_n = \frac{1}{\sqrt{n\pi}}$ ([10], Theorem 7.3), $\beta_n = 0$ ([10], Lemma 8.4), and $\mu_n, \lambda_{2n}, \lambda_{2n-1}$ all coincide and are equal to $\pi^2 n^2$. Therefore, for λ near $n^2\pi^2$, $\lambda \neq n^2\pi^2$,

$$\begin{aligned}
\Delta_n|_{q=0} =& \frac{4\lambda}{n^2\pi^2} \left(\prod_{m\geq 1} \frac{m^2\pi^2 - \lambda}{m^2\pi^2} \right)^2 \left(\frac{n^2\pi^2}{n^2\pi^2 - \lambda} \right)^2 \\
=& \frac{4\lambda}{n^2\pi^2} \left(\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} \right)^2 \frac{(n^2\pi^2)^2}{(n\pi + \sqrt{\lambda})^2 (n\pi - \sqrt{\lambda})^2} \\
=& \frac{4n^2\pi^2}{(n\pi + \sqrt{\lambda})^2} \left(\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda} - n\pi} \right)^2 .
\end{aligned}$$

As a consequence

$$\lim_{\lambda \to n^2\pi^2} \Delta_n(\lambda)|_{q=0} = 1.$$

Hence

$$\begin{aligned}
\partial_q(z_{-n})|_{q=0} =& \frac{1}{\sqrt{n\pi}} \partial_q(u_n^+/2)|_{q=0} \\
=& \frac{1}{\sqrt{n\pi}} \left(\partial_q(\tau_n - \mu_n) + i2\pi n \cosh\kappa_n \cdot \partial_q\kappa_n \right)|_{q=0}. \qquad (100)
\end{aligned}$$

Note that at $q=0$, $\kappa_n=0$ and hence $\cosh\kappa_n=1$. Furthermore

$$\partial_q\kappa_n=\frac{1}{y_2'(1,\mu_n)}\left(\partial_q y_2'(1,\mu_n)+\dot{y}_2'(1,\mu_n)\cdot\partial_q\mu_n\right)$$

At $q=0$,

$$y_2'(1,\mu_n)=\cos n\pi=(-1)^n$$

and

$$\dot{y}_2'(1,\mu_n)=\partial_\lambda\partial_x\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\Big|_{\lambda=n^2\pi^2,x=1}=0.$$

Moreover (see e.g. [10], p 195)

$$\partial_q y_2'(1,\mu_n)=y_2'(1,n^2\pi^2)\cos n\pi x\frac{\sin n\pi x}{n\pi}=\frac{(-1)^n}{2n\pi}\sin 2n\pi x$$

and thus

$$2\pi n\cosh\kappa_n\cdot\partial_q\kappa_n|_{q=0}=\sin 2n\pi x. \tag{101}$$

Next let us compute $\partial_q(\tau_n-\mu_n)$ at $q=0$. Using Riesz projectors one has, for any q near 0,

$$\mu_n=\mathrm{Tr}\left(\frac{1}{2\pi i}\int_{\Gamma_n}\lambda(\lambda-L_D(q))^{-1}d\lambda\right)$$

and

$$2\tau_n=\mathrm{Tr}\left(\frac{1}{2\pi i}\int_{\Gamma_n}\lambda(\lambda-L_p(q))^{-1}d\lambda\right)$$

where Γ_n is a counterclockwise oriented contour around $n^2\pi^2$ and $L_D(q)$ $[L_p(q)]$ is the operator $L(q)=-d_x^2+q$ considered on the space $H_0^2[0,1]$ $[H^2(\mathbb{R}/2\mathbb{Z},\mathbb{R})]$. Then

$$\langle\partial_q\mu_n,h\rangle=\mathrm{Tr}\left(\frac{1}{2\pi i}\int_{\Gamma_n}\lambda(\lambda-L_D(q))^{-1}h(\lambda-L_D(q))^{-1}d\lambda\right).$$

At $q=0$, $\sqrt{2}\sin n\pi x$ is the L^2–normalized eigenfunction corresponding to $\mu_n=n^2\pi^2$. Hence

$$\begin{aligned}\langle\partial_q\mu_n,h\rangle=&\langle\sqrt{2}\sin n\pi x,\frac{1}{2\pi i}\int_{\Gamma_n}\frac{\lambda}{(\lambda-n^2\pi^2)^2}h\sqrt{2}\sin n\pi x d\lambda\rangle\\ =&\int_0^1 h(x)\sin^2 n\pi x dx.\end{aligned}$$

Similarly at $q=0$, $\sqrt{2}\cos n\pi x$ and $\sqrt{2}\sin n\pi x$ are an orthonormal basis of the eigenspace of $L_{per}(q)$ corresponding to the periodic eigenvalue $\lambda_{2n}=\lambda_{2n-1}=n^2\pi^2$. Hence

$$\begin{aligned}\langle 2\tau_n,h\rangle=&\langle\sqrt{2}\sin n\pi x,\frac{1}{2\pi i}\int_{\Gamma_n}\frac{\lambda}{(\lambda-n^2\pi^2)^2}h\sqrt{2}\sin n\pi x d\lambda\rangle\\ &+\langle\sqrt{2}\cos n\pi x,\frac{1}{2\pi i}\int_{\Gamma_n}\frac{\lambda}{(\lambda-n^2\pi^2)^2}h\sqrt{2}\sin n\pi x d\lambda\rangle\\ =&\int_0^1 h(x)2(\sin^2 n\pi x+\cos^2 n\pi x)dx.\end{aligned}$$

As $2\sin^2 n\pi x = 1 - \cos 2n\pi x$ it then follows that

$$\partial_q(\tau_n - \mu_n) = \cos 2n\pi x. \tag{102}$$

Combining (100)-(102) then yields

$$\partial_q z_{-n}|_{q=0} = \frac{1}{\sqrt{n\pi}} e^{i2\pi nx}.$$

Similar computations lead to

$$\partial_q z_n|_{q=0} = \frac{1}{\sqrt{n\pi}} e^{-i2\pi nx}.$$

References

[1] D. Bättig, T. Kappeler, B. Mityagin : *On the Korteweg-de Vries equation: frequencies and initial value problem*, Pacific J. of Math. 181 (1997), 1-55.

[2] M. Erdogan, N. Tzirakis, V. Zharnitsky : *Near-linear dynamics in KdV with periodic boundary conditions*, Nonlinearity 23 (2010), 1675-1694.

[3] H. Flaschka, D. McLaughlin: *Canonically conjugate variables for the Korteweg-de Vries equation and Toda lattices with periodic boundary conditions*, Progress Theor.Phys. 55(1976), 438-456.

[4] B. Grébert, T. Kappeler, J. Pöschel: *Normal form theory of the NLS equation*, preliminary version, arxiv.org/abs/0907.3938v1 2009. To appear in *EMS Series of Lectures in Mathematics*, EMS Publishing House.

[5] A. Its, V. Matveev : *A class of solutions of the Korteweg-de Vries equation*(in Russian), Problems in Math. Physics N08, Izdat. Leningrad Univ., Leningrad, 1976 , 79-92.

[6] T. Kappeler: *Fibration of the phase-space for the Korteweg-de Vries equation*, Ann. Inst. Fourier 41 (1991), 539-575.

[7] T. Kappeler, M. Makarov: *On action-angle variables for the second Poisson bracket*, Comm. in Math. Phys., 214, 3 (2000), 651-677.

[8] T. Kappeler, M. Makarov: *On Birkhoff coordinates for KdV*, Ann. H. Poincaré 2 (2001), 807 - 856.

[9] T. Kappeler, B. Mityagin: *Estimates for periodic and Dirichlet eigenvalues of the Schrödinger operator*, SIAM J. Math Anal. 33 (2001), p 113-152.

[10] T. Kappeler, J. Pöschel: *KdV & KAM*, Ergebnisse Math. u. Grenzgebiete, Springer, Berlin, 2003.

[11] T. Kappeler, B. Schaad, P. Topalov: *mKdV and its Birkhoff coordinates*, Physica D: Nonlinear Phenomena, 237, 10-12 (2008), 1655-1662.

[12] T. Kappeler, B. Schaad, P. Topalov: *Asymptotic estimates of spectral quantities of Schrödinger operators*, arXiv: 1107.4542

[13] T. Kappeler, P. Topalov: *Global fold structure of the Miura map on* $L^2(\mathbb{T}, \mathbb{R})$, Int. Math. Research Notices (2004), 2039-2068.

[14] T. Kappeler, P. Topalov: *Global wellposedness of KdV in* $H^{-1}(\mathbb{T}, \mathbb{R})$, Duke Math. J. 135 (2006), 327-360.

[15] E. Korotyaev: *Estimates for the Hill operator, II*, J. Diff. Eq. 223 , 2006, 229-260.

[16] S. Kuksin: *Damped driven KdV and effective equation for long-time behaviour of its solution*, preliminary version, 2009.

[17] S. Kuksin, G. Perelman: *Vey theorem in infinite dimensions and its application to KdV*, Discrete and Continuous Dynamical Systems Ser. A, 27, no 1, (2010), 1-24.

[18] S. Kuksin, A. Piatnitski: *Khasminskii - Whitham averaging for randomly perturbated KdV equation*, J. Math. Pures Appl. 89 (2008), 400-428.

[19] V. Marchenko: *Sturm-Liouville operators and applications*, Birkhäuser, Basel, 1986.

[20] H. McKean, E. Vaninsky: *Action-angle variables for the cubic Schroedinger equation*, Comm. Pure Appl. Math. 50 (1997), 489-562.

[21] J. Pöschel, E. Trubowitz: *Inverse spectral theory*, Academic Press, Boston, 1987.

[22] A. Savchuk, A. Shkalikov: *On the eigenvalues of the Sturm-Liouville operator with potentials from Sobolev spaces*, Math. Notes 80, no 6 (2006), 864-884.

[23] B. Schaad: *Qualitative features of periodic solutions of KdV*, PhD thesis, in preparation.